Power Series with Regular Terms on Complex Banach Lattices

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Banach lattice

Let E be a Banach space.

Suppose

- E has a partial order relation $\leq$ compatible with the linear structure of E
- and every $x, y \in E$ have supremum $x \vee y$ and infimum $x \wedge y$

Define $|x| = x \vee -x$.

If $|x| \leq |y|$ implies that $\|x\| \leq \|y\|$,

then $(E, \|\cdot\|)$ is called a Banach lattice.

Example: $(C(K), \|\cdot\|_\infty)$ is a Banach lattice with the pointwise order $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in K$.

Complex Banach lattices

$E_{\mathbb{C}}$ is the complexification of a real vector space E:

- $E_{\mathbb{C}}$ is a complex vector space in which every element z can be expressed uniquely in the form $z = x + iy$, where $x, y \in E$.
- $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$ are the real and imaginary parts of z respectively.

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If E is a Banach lattice, the modulus of $z = x + iy \in E_{\mathbb{C}}$ is

$$|z| = \sqrt{x^2 + y^2} = \sup\{x \cos\theta + y \sin\theta : 0 \leq \theta \leq 2\pi\} \in E$$

defined via the Krivine functional calculus.

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Define a norm on $E_{\mathbb{C}}$ by

$$\|z\| = \| \, |z| \, \|$$

The triple $(E_{\mathbb{C}}, |\cdot|, \|\cdot\|)$ is called a complex Banach lattice.

Recall: Homogeneous polynomials

- Let E, F be (real) Banach lattices.
- Let $A\colon E^m \to F$ be an m-linear mapping.
- An *m*-homogeneous polynomial $P\colon E \to F$ is generated by a unique symmetric m-linear mapping A:

$$P(x) = A(x, \ldots, x)$$

  for every $x \in E$. Write $P = \widehat{A}$.
- P is positive if $A(x_1, \ldots, x_m) \geq 0$ for all $x_1, \ldots, x_m \geq 0$
- P is regular if it is the difference of two positive polynomials.

Homogeneous polynomials

- *Dedekind complete*: every order bounded set has a supremum.
- Let E, F be (real) Banach lattices with F Dedekind complete.
- Bu and Buskes, 2012: The space $\mathcal{P}_r(^m E; F)$ of regular m-homogeneous polynomials is a Banach lattice with the regular norm, defined by

$$\|P\|_r = \||P|\|$$

where $\|\cdot\|$ is the supremum norm.

Complexification

- Let $E_1, \ldots, E_m, F$ be real vector spaces.
- Let $A\colon E_1 \times \cdots \times E_m \to F$ be an m-linear mapping.
- A has a unique extension to a complex m-linear mapping $A_{\mathbb{C}}\colon (E_1)_{\mathbb{C}} \times \cdots \times (E_m)_{\mathbb{C}} \to F_{\mathbb{C}}$.
- For $z_j = x_j^0 + ix_j^1 \in (E_j)_{\mathbb{C}}$, $1 \leq j \leq m$, we have

$$A_{\mathbb{C}}(z_1, \ldots, z_m) = \sum_{\delta_1, \ldots, \delta_m = 0,1} i^{\sum \delta_j} A(x_1^{\delta_1}, \ldots, x_m^{\delta_m}) \,.$$

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- A complex m-linear mapping from $(E_1)_{\mathbb{C}} \times \ldots (E_m)_{\mathbb{C}}$ into $F_{\mathbb{C}}$ is said to be real if it is the complexification of a real m linear mapping from $E_1 \times \cdots \times E_m$ into F.

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- Every complex m-linear mapping A can be decomposed uniquely in the form $A = A_0 + iA_1$, where A_0, A_1 are real m-linear mappings.

Complexification

$$A = A_0 + iA_1$$

- Let

$$A_0(x_1, \ldots, x_m) = \operatorname{Re} A(x_1, \ldots, x_m)$$
$$A_1(x_1, \ldots, x_m) = \operatorname{Im} A(x_1, \ldots, x_m)$$

for real arguments and then extend A_0, A_1 to the complexification.

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  for real arguments and then extend A_0, A_1 to the complexification.

- So the vector space of complex m-linear mappings is the complexification of the space of real m-linear mappings:

$$\mathcal{L}(^m(E_1)_{\mathbb{C}}, \ldots, (E_m)_{\mathbb{C}}; F_{\mathbb{C}}) \cong \mathcal{L}(^m E_1, \ldots, E_m; F)_{\mathbb{C}} \,.$$

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- If the spaces $E_1, \ldots, E_m$ are the same then an m-linear mapping A on E^m is symmetric if and only if its complexification, $A_{\mathbb{C}}$, is.

Complexification

- If E, F are real vector spaces, then every m-homogeneous polynomial $P = \widehat{A} \colon E \to F$ has a unique extension to a complex m-homogeneous polynomial $P_{\mathbb{C}} \colon E_{\mathbb{C}} \to F_{\mathbb{C}}$.
- From now $P(z)$ will be understood to mean $P_{\mathbb{C}}(z)$.
- A complex m-homogeneous polynomial that arises as the complexification of a polynomial from E into F is called a real polynomial.
- The space of complex m-homogeneous polynomials is the complexification of the space of real m-homogeneous polynomials:
$$P({}^m E_{\mathbb{C}}; F_{\mathbb{C}}) \cong P({}^m E; F)_{\mathbb{C}}\,.$$

Complexification

- Every complex m-homogeneous polynomial P can be written uniquely in the form $P = P_0 + iP_1$, where P_0, P_1 are real m-homogeneous polynomials, defined, for real arguments by

$$P_0(x) = \operatorname{Re} P(x) \quad \text{and} \quad P_1(x) = \operatorname{Im} P(x)$$

  for $x \in E$. The polynomials P_0, P_1 are then extended by complexification to all of $E_{\mathbb{C}}$.

- Remark: the identity $(\operatorname{Re} P)(z) = \operatorname{Re}(P(z))$ is only valid for real arguments. So $P_0(z) \neq \operatorname{Re} P(z)$ in general.

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- Example: $P(z) = z^2$ on $\mathbb{C}$.

Complexification

- Let E, F be complex Banach lattices, with F Dedekind complete.
- $E_{\mathbb{R}}$ is the real part of E, i.e. E is the complexification of $E_{\mathbb{R}}$.
- Define regularity, positivity, respectively, for multilinear and polynomial mappings as the linear case:

  An m-linear mapping on E^m, or an m-homogeneous polynomial on E, is regular, positive, respectively, if both its real and imaginary parts are regular, positive on $(E_{\mathbb{R}})^m$ or $E_{\mathbb{R}}$, respectively.

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- In general

$$\|P\| \leq \|P\|_r$$

  and the inequality can be strict.

Regular norm

Proposition (Choi-Kim-Ki, 1998)

Let $a, b, c \in \mathbb{R}$ with $|a| < 1$, $|b| < 1$ and $2 < |c| \leq 4$. Suppose $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\ell_1^2)$. Then, over both the real and complex numbers,

$$\|P\| = 1 \quad \text{if and only if} \quad 4|c| - c^2 = 4(|a+b| - ab)\,.$$

Example: On ℓ_1^2, let

$$P(x, y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 + (2 + \sqrt{3})xy\,.$$

Then

$$\|P\| = 1 \quad \text{and} \quad \|P\|_r = \frac{3 + \sqrt{3}}{4} > 1\,.$$

over both the real and complex numbers.

Orthogonally additive polynomials

- $z, w \in E$ are said to be disjoint, denoted by $z \perp w$, if $|z| \wedge |w| = 0$ in $E_{\mathbb{R}}$.

- P is orthogonally additive if

$$P(z+w) = P(z) + P(w)$$

whenever $z \perp w$.

Orthogonally additive polynomials

Proposition (C. Boyd, R. Ryan and S., TAMS 2025)

Let E be a complex Banach lattice and P be a regular m-homogeneous polynomial on E.

Then the following are equivalent:

(a) *P is orthogonally additive on E.*
(b) *P is orthogonally additive on $E_{\mathbb{R}}$.*
(c) *The real and imaginary parts of P are orthogonally additive on E.*

Orthogonally additive polynomials

- C. Boyd, R. Ryan, S., 2020: If P is an m-homogeneous orthogonally additive polynomial on a **real** Banach lattice then $\|P\| = \|P\|_r$ when m is odd and $\|P\| \leq \|P\|_r \leq 2\|P\|$ when m is even.
- The polynomial $P(x) = x_1^m - x_2^m$, with m even, on ℓ_∞^2 shows that this bound is sharp.

Theorem (C. Boyd, R. Ryan and S., TAMS 2025)

Let P be an orthogonally additive m-homogeneous polynomial on a **complex** *Banach lattice E. Then $\|P\| = \|P\|_r$.*

Proof: main ingredients

- First consider P on the complex Banach lattice $C(K)$.
- Aron and Berner: every (orthogonally additive) m-homogeneous polynomial P on $C(K)$ has an extension, $\tilde{P}$, to $C(K)''$.
- Carando, Lassalle and Zalduendo: $\tilde{P}$ is orthogonally additive.
- Davie and Gamelin: $\|\tilde{P}\| = \|P\|$.

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- Carando, Lassalle and Zalduendo: $\tilde{P}$ is orthogonally additive.
- Davie and Gamelin: $\|\tilde{P}\| = \|P\|$.
- $\mathcal{B}(K)$, the space of all bounded Borel measurable functions on K is a closed subspace of $C(K)''$.
- So each orthogonally additive m-homogeneous polynomial P on $C(K)$ has an extension P_B to $\mathcal{B}(K)$ as an orthogonally additive polynomial with $\|P_B\| = \|P\|$.

Proof: main ingredients

- Carando, Lassalle and Zalduendo: Since P is orthogonally additive on $C(K)$, there is a complex measure μ on K such that

$$P(x) = \int_K x^m(t)\, d\mu(t)$$

  for all x in $C(K)$.

- Using this, we can show that $|P|$ is represented by the measure $|\mu|$.

- For all bounded Borel measurable functions y on K:

$$P_B(y) = \tilde{P}(y) = \int_K y^m(t)\, d\mu(t)$$

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- For all bounded Borel measurable functions y on K:

$$P_B(y) = \tilde{P}(y) = \int_K y^m(t)\, d\mu(t)$$

- There exists a Borel measurable function $\rho\colon K \to \mathbb{C}$ with $|\rho(t)| = 1$ for all t in K such that $d|\mu| = \rho d\mu$.

Proof: main ingredients

- Choose a branch $\rho^{1/m}$ of the m-th root of ρ. For each x in $C(K)$, $x\rho^{1/m}$ is a bounded Borel function on K. Therefore, given an orthogonally additive m-homogeneous polynomial P on $C(K)$ we have

$$|P|(x) = P_B(x\rho^{1/m}).$$

Proof: main ingredients

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$$|P|(x) = P_B(x\rho^{1/m}).$$
- Thus, we have
$$\begin{aligned}\|P\| \leq \|P\|_r &= \sup_{\|x\|\leq 1, x\in C(K)} ||P|(x)| \\ &= \sup_{\|x\|\leq 1, x\in C(K)} |P_B(x\rho^{1/m})| \\ &\leq \sup_{\|y\|\leq 1, y\in B(K)} |P_B(y)| \\ &= \|P\|.\end{aligned}$$
- So $\|P\| = \|P\|_r$.

Proof: main ingredients

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- Thus, we have

$$\begin{aligned}
\|P\| \leq \|P\|_r &= \sup_{\|x\|\leq 1, x\in C(K)} ||P|(x)| \\
&= \sup_{\|x\|\leq 1, x\in C(K)} |P_B(x\rho^{1/m})| \\
&\leq \sup_{\|y\|\leq 1, y\in B(K)} |P_B(y)| \\
&= \|P\|.
\end{aligned}$$

- So $\|P\| = \|P\|_r$.
- Extend this result to E using Kakutani Representation Theorem.

Regular norm and complexification

- The supremum norm of a polynomial on real Banach spaces is not, in general, preserved by complexification.
- Gustavo, Muñoz, Tonge 1999: If E, F are real Banach spaces and $P\colon E \to F$ is a bounded m-homogeneous polynomial, then the norm of its complexification satisfies

$$\|P_{\mathbb{C}}\|_\nu \leq 2^{m-1}\|P\|$$

  where ν is any "natural complexification process" and this inequality is sharp.
- Is the regular norm preserved by complexification?

Regular norm and complexification

Proposition (C. Boyd, R. Ryan and S., TAMS 2025)

Let E, F be real Banach lattices, with F Dedekind complete and let $P \in \mathcal{P}_r({}^m E; F)$. Then $|P_{\mathbb{C}}| = |P|_{\mathbb{C}}$.

This gives us

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Modulus

- Shipper and Schaefer (independently of each other) 1973–1974: The absolute value of a regular linear mapping $T\colon E \to F$ between complex Banach lattices satisfies

$$|T(z)| \leq |T|(|z|)$$

for every $z \in E$.

- Does it hold for polynomials?

Modulus

Proposition (Boyd, Ryan and S., 2025 TAMS)

Let E, F be real or complex Banach lattices, with F Dedekind complete. Then the mapping

$$A \in \mathcal{L}_r({}^m E; F) \mapsto A^{(k)} \in \mathcal{L}_r({}^{m-k} E; \mathcal{L}_r({}^k E; F))$$

is a Banach lattice isometric isomorphism for every $k = 1, \ldots, m-1$.

This proposition combined with an inductive argument gives us

Proposition (Boyd, Ryan and S., TAMS 2025)

Let E, F be complex Banach lattices, with F Dedekind complete and let $P\colon E \to F$ be a regular m-homogeneous polynomial. Then

$$|P(z)| \leq |P|(|z|)$$

for all $z \in E$.

Power series with regular terms

A power series $\sum_m P_m$ with regular terms on a complex Banach lattice E is regularly convergent at a point $z \in E$ if the series $\sum_m |P_m|(|z|)$ converges.

This is stronger than absolute convergence, since

$$|P_m(z)| \leq |P_m|(|z|)$$

for all $z \in E$.

Radius of regular convergence

- For a power series $f = \sum_m P_m(z - a)$ about the point a on a complex Banach space, the radius of convergence:

$$r(f, a) = \left(\limsup \|P_m\|^{1/m}\right)^{-1}.$$

  This number is the supremum of the set of nonnegative real numbers r for which the series is uniformly convergent on the closed ball of radius r centred at a.

- We define the radius of regular convergence of a power series $f = \sum_m P_m(z - a)$ about the point a to be

$$|r|(f, a) = \left(\limsup \|P_m\|_r{}^{1/m}\right)^{-1}.$$

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This is the supremum of the set of nonnegative real numbers ρ for which the series is uniformly regularly convergent in the closed ball of radius ρ centred at a.

- Since $\|P_m\| \leq \|P_m\|_r$, we have $|r|(f,a) \leq r(f,a)$.

Radius of regular convergence

Theorem (Boyd, Ryan and S., TAMS 2025)

Let f be an orthogonally additive holomorphic function on a complex Banach lattice E. Then for each $a \in E$ with $\|a\| < r(f, 0)$ we have

$$|r|(f, a) = r(f, a).$$

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In general, can we have a strict inequality

$$|r|(f, a) < r(f, a)?$$

Regular holomorphic functions

Let E, F be complex Banach lattices, with F Dedekind complete. Let U be an open subset of E. A function $f\colon U \to F$ is regularly holomorphic if

(a) f is holomorphic on U.

(b) For every $z \in U$, the derivatives $\frac{1}{m!}\widehat{d^m}f(z)$ are regular m-homogeneous polynomials.

(c) For every $z \in U$, the Taylor series of f at z is regularly convergent in some neighbourhood of z.

Condition (c) above is equivalent to

$$\limsup_m \left\| \frac{1}{m!}\widehat{d^m}f(z) \right\|_r^{\frac{1}{m}} < \infty$$

for every $z \in U$.

A regular holomorphic function $f\colon E \to \mathbb{C}$ is said to be orthogonally additive if $f(z+w) = f(z) + f(w)$ whenever z and w are disjoint.