

# Optimal Control of Nematic Liquid Crystals

J. P. Borgna, P. Panayotaros, D. Rial, C. S. de la Vega

Di Tella Workshop on Analysis and Beyond

18 y 19 de junio de 2025

## 1 Introduction

- The motivation
- Experimental device
- The mathematical models

## 2 Previous Results

- Linear model
- Pre-tilted nematicon model
- Full nematicon model

## 3 Optimal Control model

- Well posedness and regularity
- Existence of minimum
- First order necessary conditions
- Characterization of an optimal control

## 4 Ongoing work

- Numerical approximation

## 5 Future work

# Schrödinger equation

- Free Schrödinger:

$$iu_z + \nabla_x^2 u = 0, \quad u(0, x) = u_0(x)$$

has a dispersive effect

$$\|u(z)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(2\pi z)^{n/2}} \|u_0\|_{L^1(\mathbb{R}^n)}.$$

- Cubic nonlinear Schrödinger equation:

$$iu_z + \frac{1}{2} \nabla_x^2 u + |u|^2 u = 0, \quad u(0, x) = u_0(x)$$

has solitary waves (localized, with enduring and persistent shape) for  $n = 1$  with intensity  $|u|$  constant.

- For  $n \geq 2$  and  $\|u_0\|_{L^2} \geq a$ , solutions blow-up. Thus, there are not spatial solitary waves.

# Schrödinger equation

- Free Schrödinger:

$$iu_z + \nabla_x^2 u = 0, \quad u(0, x) = u_0(x)$$

has a dispersive effect

$$\|u(z)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(2\pi z)^{n/2}} \|u_0\|_{L^1(\mathbb{R}^n)}.$$

- Cubic nonlinear Schrödinger equation:

$$iu_z + \frac{1}{2} \nabla_x^2 u + |u|^2 u = 0, \quad u(0, x) = u_0(x)$$

has solitary waves (localized, with enduring and persistent shape) for  $n = 1$  with intensity  $|u|$  constant.

- For  $n \geq 2$  and  $\|u_0\|_{L^2} \geq a$ , solutions blow-up. Thus, there are not spatial solitary waves.

## Schrödinger equation

- Free Schrödinger:

$$iu_z + \nabla_x^2 u = 0, \quad u(0, x) = u_0(x)$$

has a dispersive effect

$$\|u(z)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(2\pi z)^{n/2}} \|u_0\|_{L^1(\mathbb{R}^n)}.$$

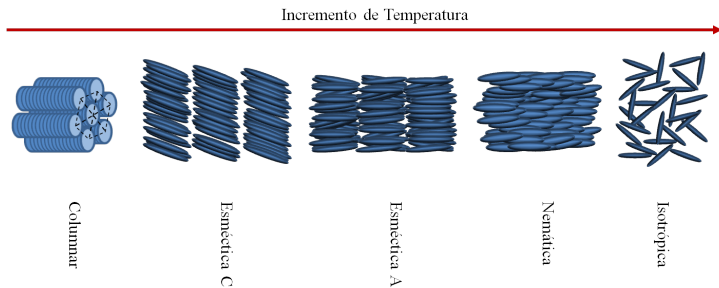
- Cubic nonlinear Schrödinger equation:

$$iu_z + \frac{1}{2} \nabla_x^2 u + |u|^2 u = 0, \quad u(0, x) = u_0(x)$$

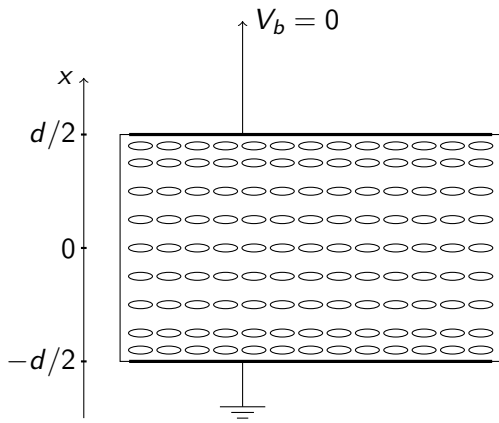
has solitary waves (localized, with enduring and persistent shape) for  $n = 1$  with intensity  $|u|$  constant.

- For  $n \geq 2$  and  $\|u_0\|_{L^2} \geq a$ , solutions blow-up. Thus, there are not spatial solitary waves.

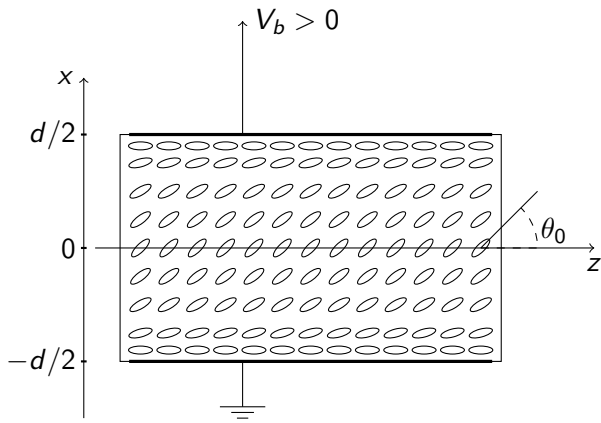
# Nematic Liquid Crystals



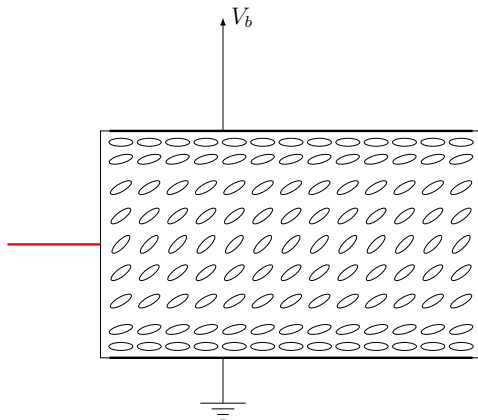
# Scheme of the experimental device



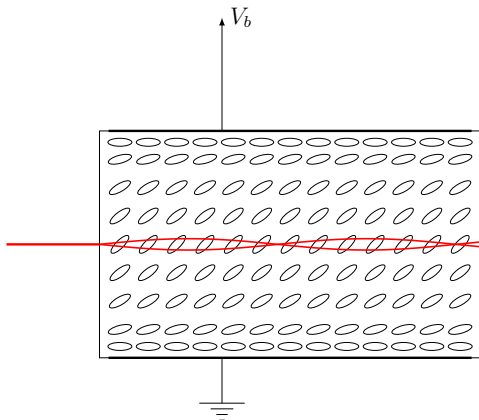
## Scheme of the experimental device



## Scheme of the experimental device



## Scheme of the experimental device



# Full Nematicon Model

$\theta_0$ : pretilt angle produced by the constant external electric field  $E_b \hat{x}$

$u$ : electric field amplitud of the laser beam

$\theta$ : additional angle produced by the laser beam

$$\begin{cases} iu_z = -\frac{1}{2}\nabla_x^2 u - \gamma(\sin^2(\theta_0 + \theta) - \sin^2(\theta_0))u, \\ \nu\nabla_x^2 \theta = \frac{1}{2}E_b^2 \sin(2\theta_0) - \frac{1}{2}(E_b^2 + |u|^2) \sin(2(\theta_0 + \theta)). \end{cases} \quad (1)$$

where  $\nu, \gamma, E_b$  are positive constants and  $x \in \mathbb{R}^2, z \geq 0$ .

Peccianti, M., De Rossi, A., Assanto, G., De Luca, A., Umeton, C. and Khoo IC. Electrically assisted self-confinement and waveguiding in planar nematic liquid crystal cells. *Applied Physics Letters*. 2000.

# Pretilt nematicon model

Assuming  $\theta_0 \sim \pi/4$ ,  $|\theta| \ll 1$ ,  $q > 0$  depending on  $\theta_0$ , we have the simpler nonlinear model:

$$\begin{cases} iu_z + \frac{1}{2}\nabla^2 u + \sin(2\theta)u = 0, \\ \nu\nabla^2\theta - q\sin(2\theta) + 2|u|^2\cos(2\theta) = 0. \end{cases} \quad (2)$$

where  $q, \nu$  are positive constants,  $x \in \mathbb{R}^2$ .

# The linear model

Considering  $|\theta| \ll 1$ , then  $\cos(2\theta) \approx 1$  and  $\sin(2\theta) \approx 2\theta$ .

$$\begin{cases} iu_z + \frac{1}{2}\nabla^2 u + 2\theta u = 0, \\ \nu\nabla^2\theta - 2q\theta + 2|u|^2 = 0, \end{cases} \quad (3)$$

where  $q, \nu$  are positive constants,  $x \in \mathbb{R}^2$ .

## 1 Introduction

- The motivation
- Experimental device
- The mathematical models

## 2 Previous Results

- Linear model
- Pre-tilted nematicon model
- Full nematicon model

## 3 Optimal Control model

- Well posedness and regularity
- Existence of minimum
- First order necessary conditions
- Characterization of an optimal control

## 4 Ongoing work

- Numerical approximation

## 5 Future work

## Nematicon lineal model

Panayotaros, P. and Marchant, TR. Solitary waves in nematic liquid crystals. *Physica D: Nonlinear Phenomena*. 2014.

$$\begin{cases} iu_z + \frac{1}{2}\nabla^2 u + 2\theta u = 0, \\ \nu\nabla^2\theta - 2q\theta + 2|u|^2 = 0, \end{cases}$$

where  $q, \nu$  are positive constants,  $x \in \mathbb{R}^2$ .

## Results

- Equation on  $\theta$  can be solved

$$\theta(u) = G(|u|^2)(x) = \frac{2}{\nu} \int_{\mathbb{R}^2} N_0(\sqrt{2q/\nu}(x-y)) |u(y)|^2 dy$$

$N_0$ : Modified Bessel function.

- Equation on  $u$  satisfies

$$iu_z + \frac{1}{2} \nabla^2 u + 2G(|u|^2)u = 0, \quad u(0) = u_0$$

- Well posedness (global existence, uniqueness and continuous dependence):

$$u \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2)).$$

- Existence of the stationary solutions  $u(x, z) = e^{-i\omega z} \phi(x)$ , with  $\|\phi\|_{L^2} > a_0$ .

## Results

- Equation on  $\theta$  can be solved

$$\theta(u) = G(|u|^2)(x) = \frac{2}{\nu} \int_{\mathbb{R}^2} N_0(\sqrt{2q/\nu}(x-y)) |u(y)|^2 dy$$

$N_0$ : Modified Bessel function.

- Equation on  $u$  satisfies

$$iu_z + \frac{1}{2} \nabla^2 u + 2G(|u|^2)u = 0, \quad u(0) = u_0$$

- Well posedness (global existence, uniqueness and continuous dependence):

$$u \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2)).$$

- Existence of the stationary solutions  $u(x, z) = e^{-i\omega z} \phi(x)$ , with  $\|\phi\|_{L^2} > a_0$ .

## Results

- Equation on  $\theta$  can be solved

$$\theta(u) = G(|u|^2)(x) = \frac{2}{\nu} \int_{\mathbb{R}^2} N_0(\sqrt{2q/\nu}(x-y)) |u(y)|^2 dy$$

$N_0$ : Modified Bessel function.

- Equation on  $u$  satisfies

$$iu_z + \frac{1}{2} \nabla^2 u + 2G(|u|^2)u = 0, \quad u(0) = u_0$$

- Well posedness (global existence, uniqueness and continuous dependence):

$$u \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2)).$$

- Existence of the stationary solutions  $u(x, z) = e^{-i\omega z} \phi(x)$ ,  
with  $\|\phi\|_{L^2} > a_0$ .

## Results

- Equation on  $\theta$  can be solved

$$\theta(u) = G(|u|^2)(x) = \frac{2}{\nu} \int_{\mathbb{R}^2} N_0(\sqrt{2q/\nu}(x-y)) |u(y)|^2 dy$$

$N_0$ : Modified Bessel function.

- Equation on  $u$  satisfies

$$iu_z + \frac{1}{2} \nabla^2 u + 2G(|u|^2)u = 0, \quad u(0) = u_0$$

- Well posedness (global existence, uniqueness and continuous dependence):

$$u \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2)).$$

- Existence of the stationary solutions  $u(x, z) = e^{-i\omega z} \phi(x)$ , with  $\|\phi\|_{L^2} > a_0$ .

## Pre-tilted nematicon model

Borgna, J.P. and Panayotaros, P. and Rial, D and Sánchez de la Vega, C. Optical solitons in nematic liquid crystals: model with saturation effects. *Nonlinearity*. 2018.

We analyze the nonlinear system (2):

$$\begin{cases} iu_z + \frac{1}{2}\nabla^2 u + \sin(2\theta)u = 0, \\ \nu\nabla^2\theta - q\sin(2\theta) + 2|u|^2\cos(2\theta) = 0, \end{cases}$$

$$u(x, z = 0) = u_0(x) \text{ for } x \in \mathbb{R}^2,$$

assuming  $\theta_0 \sim \pi/4$ ,  $|\theta| \ll 1$ ,  $q > 0$  depending on  $\theta_0$ .

## Results

- Director angle equation: We prove the existence of  $\theta = \theta(u)$  and uniqueness of director angle equation.
- Replacing  $\theta(u)$  into the nonlinear Schrödinger equation, we solve the following initial value problem:

$$iu_z + \frac{1}{2}\nabla^2 u + \sin(2\theta(u))u = 0, \quad u(x, z = 0) = u_0(x).$$

- Well posedness.
- Existence of stationary solutions for laser power above a threshold.
- Saturation effect:  $0 \leq \theta \leq \pi/4$  and if  $\|u\|_{L^\infty} \rightarrow \infty$ , then

$$\theta \rightarrow \pi/4.$$

## Full nematicon model

Borgna, J.P.; Panayotaros, P.; Rial, D and Sánchez de la Vega, C. Optical solitons in nematic liquid crystals: arbitrary deviation angle model. *Physica D: Nonlinear Phenomena*. 2020.

$$\begin{cases} iu_z = -\frac{1}{2}\nabla^2 u - \gamma(\sin^2(\theta_0 + \theta) - \sin^2(\theta_0))u, \\ \nu\nabla^2\theta = \frac{1}{2}E_b^2 \sin(2\theta_0) - \frac{1}{2}(E_b^2 + |u|^2) \sin(2(\theta_0 + \theta)). \end{cases}$$

$$u(x, z = 0) = u_0(x) \text{ for } x \in \mathbb{R}^2,$$

$E_b > 0$ ,  $\theta_0 \in (\pi/4, \pi/2)$  and no assumptions on the size of  $\theta$ .

# Results

- Well posedness.
- Existence of stationary solutions.
- Saturation effect:  $0 \leq \theta \leq \pi/2 - \theta_0$  and if  $\|u\|_{L^\infty} \rightarrow \infty$ , then

$$\theta + \theta_0 \rightarrow \pi/2.$$

## 1 Introduction

- The motivation
- Experimental device
- The mathematical models

## 2 Previous Results

- Linear model
- Pre-tilted nematicon model
- Full nematicon model

## 3 Optimal Control model

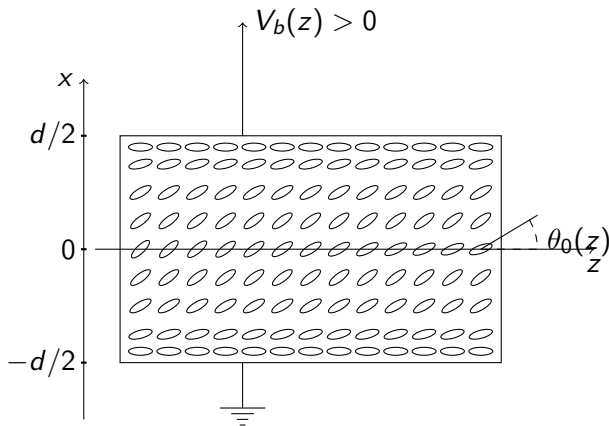
- Well posedness and regularity
- Existence of minimum
- First order necessary conditions
- Characterization of an optimal control

## 4 Ongoing work

- Numerical approximation

## 5 Future work

## The device



## The optimal control model

Borgna, J.P.; Rial, D and Sánchez de la Vega, C. Optimal control for optical solitons in nematic liquid crystals. *Applied Mathematics and Optimization*. 2024.

We analyze the optimal control problem

$$\inf_{q \in \mathcal{Q}_{\text{ad}}} \mathcal{J}(q) = \|u[q](\zeta) - u_1\|_{L^2(\mathbb{R}^2)}^2 + \kappa \|\dot{q}\|_{L^2([0, \zeta])}^2$$

where  $(q, \theta, u)$  satisfy the system:

$$\begin{cases} iu_z + \frac{1}{2}\nabla^2 u + 2\theta u = 0, \\ \nu \nabla^2 \theta - 2q(z)\theta + 2|u|^2 = 0, \end{cases} \quad (4)$$

$u(x, z = 0) = u_0(x)$  for  $x \in \mathbb{R}^2$  and

$\mathcal{Q}_{\text{ad}} = \{q \in H^1([0, \zeta]) : q(0) = q_0, q(\zeta) = q_\zeta \text{ and } m \leq q(z) \leq M \text{ for all } z\}$ .

# Results

We proved:

- Well posedness and regularity.
- Existence of a minimum.
- First order necessary conditions.

## Well posedness

- We define  $\Theta : [m, M] \times H^1(\mathbb{R}^2) \rightarrow H^2(\mathbb{R}^2)$ , where  $\theta = \Theta(q, u)$  is the unique solution of the elliptic equation:

$$\nu \nabla^2 \theta - 2q\theta + 2|u|^2 = 0.$$

- $\Theta$  is locally Lipschitz continuous.
- Given  $q \in \mathcal{Q}_{ad}$ ,  $u \in C([0, \zeta], H^1(\mathbb{R}^2))$ , we have

$$\theta(z) = \Theta(q(z), u(z)) \in C([0, \zeta], H^2(\mathbb{R}^2)).$$

- For the initial value problem with  $u(0) = u_0 \in H^1(\mathbb{R}^2)$

$$iu_z + \frac{1}{2} \nabla^2 u + 2\Theta(q, u)u = 0,$$

we prove the existence of a unique solution globally defined.

# Well posedness

## Theorem

Given  $q \in \mathcal{Q}_{ad}$ , the initial value problem for

$$\begin{cases} iu_z + \frac{1}{2}\nabla^2 u + 2\theta u = 0, \\ \nu \nabla^2 \theta - 2q(z)\theta + 2|u|^2 = 0, \end{cases}$$

with  $u(0) = u_0 \in H^1(\mathbb{R}^2)$  has a unique solution

$$u = u[q, u_0] \in C([0, \zeta], H^1(\mathbb{R}^2)) \cap C^1([0, \zeta], H^{-1}(\mathbb{R}^2))$$

$$\theta = \theta[q, u_0] \in C([0, \zeta], H^2(\mathbb{R})).$$

Moreover,  $u$  is locally Lipschitz continuous. Given  $R$  and  $\|u_0\|, \|\tilde{u}_0\| \leq R$  and  $q, \tilde{q} \in \mathcal{Q}_{ad}$ , there exists  $M = M(R, \zeta, m) > 0$ :

$$\|u - \tilde{u}\|_{C([0, \zeta], H^1(\mathbb{R}^2))} \leq M (\|u_0 - \tilde{u}_0\|_{H^1(\mathbb{R}^2)} + \|q - \tilde{q}\|_{C([0, \zeta])}).$$

Also,  $\|\theta\|_{C([0, \zeta], H^2)} \leq C \|u\|_{C([0, \zeta], H^1)}^2$ .

## Regularity of the solution

### Theorem

$\Theta(q, u)$  is Fréchet differentiable with respect to  $q$  and  $u$ .

### Theorem

Given  $u_0 \in H^1(\mathbb{R}^2)$  and  $q \in \mathcal{Q}_{ad}$ , let  $\theta \in C([0, \zeta], H^2(\mathbb{R}^2))$  and  $u \in C([0, \zeta], H^1(\mathbb{R}^2))$  solution of the coupled system. Then  $u$  is Fréchet differentiable and

$$D_q u[q](\delta q) \in C([0, \zeta], H^1(\mathbb{R}^2)) \cap C^1([0, \zeta], H^{-1}(\mathbb{R}^2)).$$

In particular  $D_q u[q](\delta q)(\zeta) \in H^1(\mathbb{R}^2)$ .

## Existence of minimum

### Theorem

*There exists an optimal control  $q^* \in \mathcal{Q}_{ad}$ .*

## First order necessary conditions

$\mathcal{Q}_{ad} = \{q \in H^1([0, \zeta]) : q(0) = q_0, q(\zeta) = q_\zeta, m \leq q(z) \leq M \forall z\}$   
is a closed convex set in  $H^1([0, \zeta])$  and therefore in  $L^2([0, \zeta])$ .

Then, if  $q^* \in \mathcal{Q}_{ad}$  is a minimum, we have the Fermat's Rule

$$\langle D_q \mathcal{J}(q^*), q - q^* \rangle_{H^{-1}[0, \zeta] \times H^1[0, \zeta]} \geq 0, \text{ for all } q \in \mathcal{Q}_{ad}.$$

# First order necessary conditions

Recall

$$\mathcal{J}(q) = \|u[q](\zeta) - u_1\|_{L^2(\mathbb{R}^2)}^2 + \kappa \|\dot{q}\|_{L^2[0,\zeta]}^2.$$

We can prove that  $\mathcal{J}$  is Fréchet differentiable and

$$\begin{aligned} D_q \mathcal{J}(q)(\delta q) &= 2 \langle u[q](\zeta) - u_1, D_q u[q](\zeta)(\delta q) \rangle_{L^2(\mathbb{R}^2)} + 2\kappa \langle \dot{q}, \delta \dot{q} \rangle_{L^2[0,\zeta]} \\ &= 2 \langle (D_q u[q](\zeta))^*(u[q](\zeta) - u_1), \delta q \rangle_{L^2[0,\zeta]} + 2\kappa \langle \dot{q}, \delta \dot{q} \rangle_{L^2[0,\zeta]} \\ &= 2 \left( (D_q u[q](\zeta))^*(u[q](\zeta) - u_1) - \kappa \frac{d}{dt} \dot{q} \right) (\delta q) \end{aligned}$$

## Adjoint system

We define

$$\begin{aligned} D_q u[q](\cdot)(\zeta) &: H^1([0, \zeta]) \longrightarrow H^1(\mathbb{R}^2) \\ \delta q &\longrightarrow D_q u[q](\delta q)(\zeta) \end{aligned}$$

and prove that for  $p_\zeta \in H^1(\mathbb{R}^2)$

$$\langle p_\zeta, D_q u[q](\delta q)(\zeta) \rangle_{H^{-1}(\mathbb{R}^2) \times H^1(\mathbb{R}^2)} = \langle -\langle y, \theta \rangle_{L^2(\mathbb{R}^2)}, \delta q \rangle_{H^{-1}[0, \zeta] \times H^1[0, \zeta]}$$

where

$$\begin{aligned} p_z &= i \frac{1}{2} \nabla^2 p(z) + 2\theta p(z) - 2uy \\ p(\zeta) &= p_\zeta \\ -\nu \nabla^2 y + qy &= -2\text{Re}(\bar{u}p) \end{aligned}$$

characterizing the dual operator for  $p_\zeta \in H^1(\mathbb{R}^2)$ .

$$[D_q u[q](\cdot)(\zeta)]^* (p_\zeta) = -\langle y, \theta \rangle_{L^2(\mathbb{R}^2)} \in L^2[0, \zeta].$$

# First order necessary conditions

## Theorem

Let  $q^* \in \mathcal{Q}_{ad}$  be an optimal solution and  $u^* = u[q^*], \theta^* = \theta[q^*]$ . Then,  $p \in C([0, \zeta], H^1(\mathbb{R}^2)) \cap C^1([0, \zeta], H^{-1}(\mathbb{R}^2))$ ,  $y \in C([0, \zeta], H^2(\mathbb{R}^2))$ :

$$iu_z^* = -\frac{1}{2}\nabla^2 u^* - 2u^*\theta^*,$$

$$\nu\nabla^2\theta^* - q^*\theta^* = -|u^*|^2$$

$$u^*(0) = u_0$$

$$ip_z = -\frac{1}{2}\nabla^2 p(z) + 2ip\theta^* + 2iu^*y$$

$$\nu\nabla^2 y - q^*y = 2\text{Re}(\bar{u}^*p)$$

$$p(\zeta) = u^*(\zeta) - u_1$$

$$\left( \langle y, \theta^* \rangle_{L^2} + \kappa \frac{d}{dz} \dot{q}^* \right) (\delta q) \leq 0 \text{ for all } q \in \mathcal{Q}_{ad}$$

where the last inequality is in the distributional sense.

## Characterization of an optimal control

Let  $q^* \in \mathcal{Q}_{ad}$  be an optimal control and  $\theta^*$  and  $y$  solution of the elliptic equations. We call  $g = \frac{1}{\kappa} \langle y, \theta^* \rangle_{L^2(\mathbb{R}^2)} \in C([0, \zeta])$ . Then

$$T(\delta q) = \int_0^\zeta \left( -\dot{q}^*(z) \delta \dot{q}(z) + g(z) \delta q(z) \right) dz \leq 0,$$

for all  $\delta q \in H^1([0, \zeta])$  such that  $q = q^* + \delta q \in \mathcal{Q}_{ad}$ . We set

$$\mathcal{A}(q) = \{z \in [0, \zeta] : m < q(z) < M\} = [0, \zeta] \cap \bigcup_{i \in I} (a_i, b_i)$$

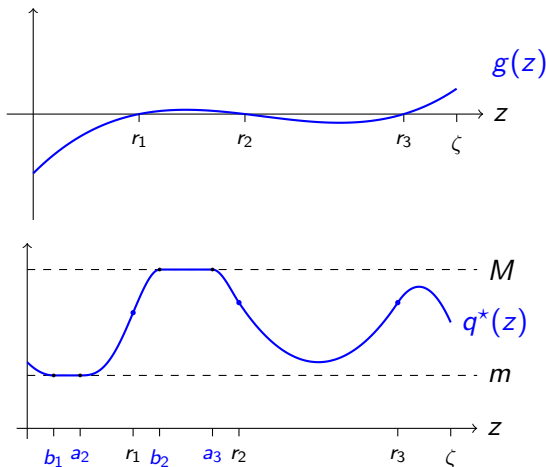
for  $I$  countable. Then

- ①  $q^*$  is unique and is the solution of

$$\inf_{q \in \mathcal{Q}_{ad}} \mathcal{J}(q) = \int_0^\zeta \left( \frac{1}{2} \dot{q}^2(z) - g(z)q(z) \right) dz.$$

- ②  $q^* \in C^2(\mathcal{A}(q^*))$  and verifies  $\ddot{q}^*(z) + g(z) = 0$  for  $z \in \mathcal{A}(q^*)$ .  
 ③  $q^* \in C^1([0, \zeta])$  and  $\dot{q}^*(z) = 0$  for all  $z \in [0, \zeta] \setminus \mathcal{A}(q^*)$ .

$$\ddot{q}^*(z) = -g(z) \text{ for } z \in \{z \in [0, \zeta] : m < q^*(z) < M\}.$$



## 1 Introduction

- The motivation
- Experimental device
- The mathematical models

## 2 Previous Results

- Linear model
- Pre-tilted nematicon model
- Full nematicon model

## 3 Optimal Control model

- Well posedness and regularity
- Existence of minimum
- First order necessary conditions
- Characterization of an optimal control

## 4 Ongoing work

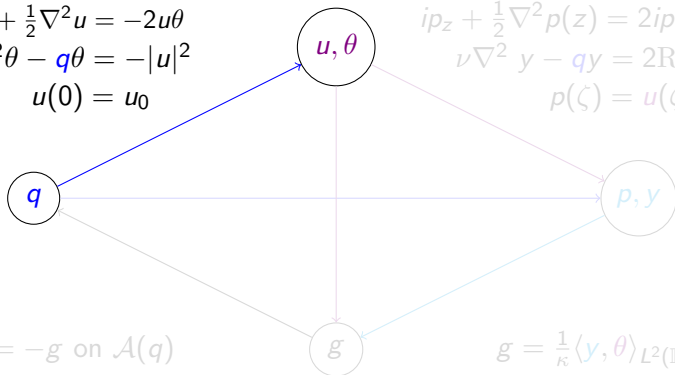
- Numerical approximation

## 5 Future work

# Numerical approximation

$$\begin{aligned}iu_z + \frac{1}{2}\nabla^2 u &= -2u\theta \\ \nu\nabla^2\theta - q\theta &= -|u|^2 \\ u(0) &= u_0\end{aligned}$$

$$\begin{aligned}ip_z + \frac{1}{2}\nabla^2 p(z) &= 2ip\theta + 2iuy \\ \nu\nabla^2 y - qy &= 2\text{Re}(\bar{u}p) \\ p(\zeta) &= u(\zeta) - u_1\end{aligned}$$



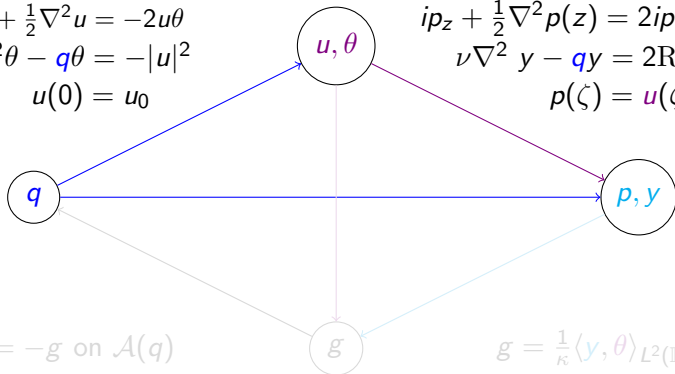
$$\ddot{q} = -g \text{ on } \mathcal{A}(q)$$

$$g = \frac{1}{\kappa} \langle y, \theta \rangle_{L^2(\mathbb{R}^2)}$$

# Numerical approximation

$$\begin{aligned}iu_z + \frac{1}{2}\nabla^2 u &= -2u\theta \\ \nu\nabla^2\theta - q\theta &= -|u|^2 \\ u(0) &= u_0\end{aligned}$$

$$\begin{aligned}ip_z + \frac{1}{2}\nabla^2 p(z) &= 2ip\theta + 2iuy \\ \nu\nabla^2 y - qy &= 2\text{Re}(\bar{u}p) \\ p(\zeta) &= u(\zeta) - u_1\end{aligned}$$



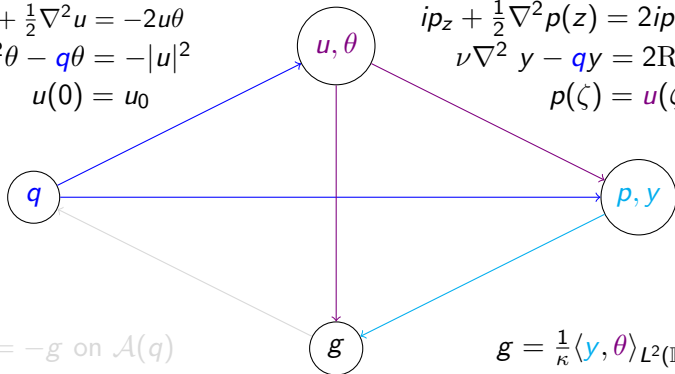
$$\ddot{q} = -g \text{ on } \mathcal{A}(q)$$

$$g = \frac{1}{\kappa} \langle y, \theta \rangle_{L^2(\mathbb{R}^2)}$$

# Numerical approximation

$$\begin{aligned} iu_z + \frac{1}{2}\nabla^2 u &= -2u\theta \\ \nu\nabla^2\theta - q\theta &= -|u|^2 \\ u(0) &= u_0 \end{aligned}$$

$$\begin{aligned} ip_z + \frac{1}{2}\nabla^2 p(z) &= 2ip\theta + 2iuy \\ \nu\nabla^2 y - qy &= 2\text{Re}(\bar{u}p) \\ p(\zeta) &= u(\zeta) - u_1 \end{aligned}$$



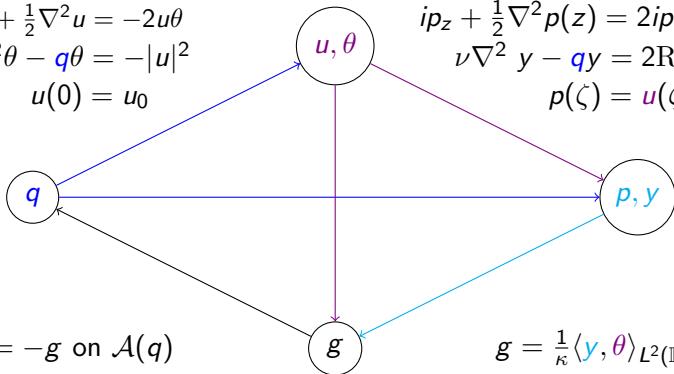
$$\ddot{q} = -g \text{ on } \mathcal{A}(q)$$

$$g = \frac{1}{\kappa} \langle y, \theta \rangle_{L^2(\mathbb{R}^2)}$$

# Numerical approximation

$$\begin{aligned}iu_z + \frac{1}{2}\nabla^2 u &= -2u\theta \\ \nu\nabla^2\theta - q\theta &= -|u|^2 \\ u(0) &= u_0\end{aligned}$$

$$\begin{aligned}ip_z + \frac{1}{2}\nabla^2 p(z) &= 2ip\theta + 2iuy \\ \nu\nabla^2 y - qy &= 2\text{Re}(\bar{u}p) \\ p(\zeta) &= u(\zeta) - u_1\end{aligned}$$



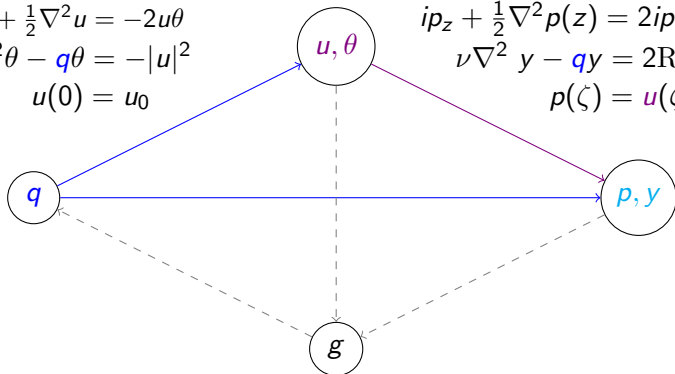
$$\ddot{q} = -g \text{ on } \mathcal{A}(q)$$

$$g = \frac{1}{\kappa} \langle y, \theta \rangle_{L^2(\mathbb{R}^2)}$$

# Solutions for Schrödinger-Poisson systems

$$\begin{aligned} iu_z + \frac{1}{2}\nabla^2 u &= -2u\theta \\ \nu\nabla^2\theta - q\theta &= -|u|^2 \\ u(0) &= u_0 \end{aligned}$$

$$\begin{aligned} ip_z + \frac{1}{2}\nabla^2 p(z) &= 2ip\theta + 2iuy \\ \nu\nabla^2 y - qy &= 2\text{Re}(\bar{u}p) \\ p(\zeta) &= u(\zeta) - u_1 \end{aligned}$$



# Spectral Runge Kutta

We consider the general Schrödinger-Poisson system

$$i v_z + \frac{1}{2} \nabla_x^2 v = \Gamma(z, v, \psi),$$

$$\nu \nabla_x^2 \psi - q \psi = \Omega(z, v),$$

where  $\mathbf{x} \in \mathbb{R}^2$  and  $z \in [0, \zeta]$ .

We consider

$$\hat{v} = \mathcal{F}(v) \text{ and } \hat{\psi} = \mathcal{F}(\psi)$$

and obtain

$$i \hat{v}_z(\xi, z) - \frac{1}{2} \sigma(\xi) \hat{v}(\xi, z) = \mathcal{F}(\Gamma(z, \mathcal{F}^{-1} \hat{v}, \mathcal{F}^{-1} \hat{\psi}))$$

$$-\nu \sigma(\xi) \hat{\psi}(\xi, z) - q(z) \hat{\psi}(\xi, z) = \mathcal{F}(\Omega(z, \mathcal{F}^{-1} \hat{v}))$$

where  $\sigma(\xi) = |\xi|^2$ .

From the equation for  $\hat{\psi}$ , we obtain

$$\hat{\psi} = -\frac{\mathcal{F}(\Omega(z, \mathcal{F}^{-1}\hat{v}))}{\nu\sigma + q(z)}$$

and from the equation for  $\hat{v}$ , the system is reduced to equation

$$i \hat{v}_z(\xi, z) - \frac{1}{2} \sigma(\xi) \hat{v}(\xi, z) = S(z, \hat{v}(\xi, z)).$$

where

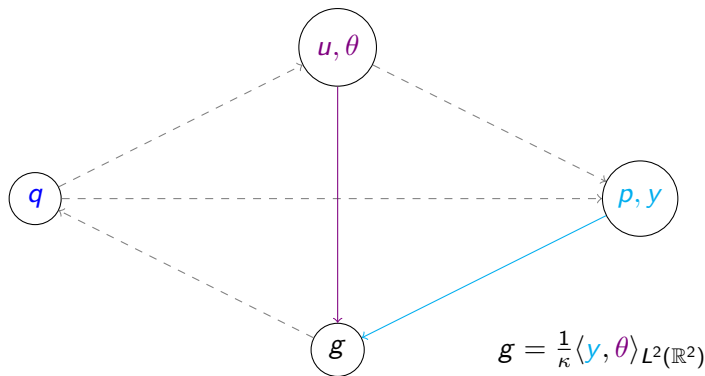
$$S(z, \hat{v}) = \mathcal{F} \left( \Gamma \left( z, \mathcal{F}^{-1}\hat{v}, \mathcal{F}^{-1} \left( -\frac{\mathcal{F}(\Omega(z, \mathcal{F}^{-1}\hat{v}))}{\nu\sigma + q(z)} \right) \right) \right).$$

Finally, we define  $w = e^{\frac{i}{2} z \sigma} \hat{v}$ , we obtain

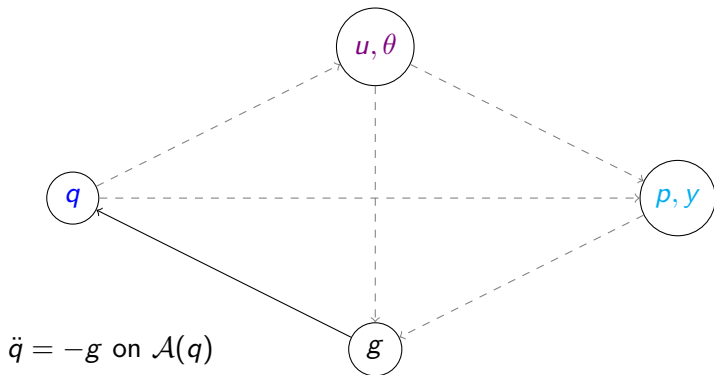
$$w_z = R(z, w)$$

where  $R(z, w) = -ie^{\frac{i}{2} z \sigma} S(z, e^{-\frac{i}{2} z \sigma} w)$ .

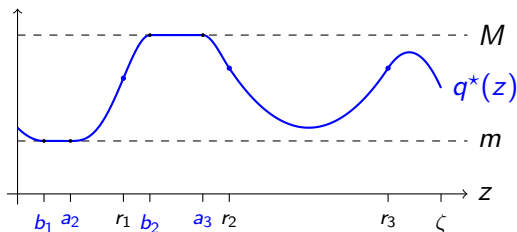
# A quadrature



## Numerical solution for the control



## Numerical solution for the control



Given  $g$ :

$$\ddot{q}(z) = -g(z) \text{ on } [0, b_1) \cup (a_2, b_2) \cup (a_3, \zeta]$$

- If there exists  $b_1$  such that

$$q(0) = q_0, q(b_1) = m, \dot{q}(b_1) = 0 \rightsquigarrow b_1 \rightsquigarrow \dot{q}_0 = \int_0^{b_1} g(z') dz'$$

- If there exists  $a_2$  and  $b_2$  such that

$$q(a_2) = m, \dot{q}(a_2) = 0, q(b_2) = M, \dot{q}(b_2) = 0 \rightsquigarrow a_2, b_2.$$

- Until the last interval  $[a_n, \zeta]$ .

## 1 Introduction

- The motivation
- Experimental device
- The mathematical models

## 2 Previous Results

- Linear model
- Pre-tilted nematicon model
- Full nematicon model

## 3 Optimal Control model

- Well posedness and regularity
- Existence of minimum
- First order necessary conditions
- Characterization of an optimal control

## 4 Ongoing work

- Numerical approximation

## 5 Future work

## Future work

- Finish the numerical simulations.
- Generalization to the more complete models.
- Controllability.

# Gracias!