# Selfimproving phenomena: Poincaré-Sobolev inequalities and BMO estimates

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# Outline

What?

Why?

How?

# Outline

What?

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^q w\right)^{\frac{1}{q}} \leq C_w \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

Why?

$$\operatorname{div}(A(x)\nabla u) = 0 \qquad , \qquad A(x)\xi.\xi \approx |\xi|^2 w(x)$$

How?

*Unweighted* $L^1$ inequalities involving "Self-improving functionals"

$$\fint_Q |f - f_Q| dx \leq a(Q)$$

# Main problem

We will study Poincaré-Sobolev type inequalities

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^q w\right)^{\frac{1}{q}} \leq C_w \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

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- For a given $\boldsymbol{p} \geq 1$.

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- There is a natural choice for a class $A_p$ of weights.

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- We try to reach the best possible $q = p_w^*$.
- Keeping track of the constant $C_w$!
- With more general "control operators" on the RHS.
- Extension to other scenarios and inequalities by means of the same method of proof.

# Unweighted Poincaré in $(\mathbb{R}^n, dx)$ - Classical results

$(1,1)$ Poincaré inequality

$$\frac{1}{|Q|}\int_Q |f - f_Q|dx \lesssim \ell(Q)\frac{1}{|Q|}\int_Q |\nabla f|dx$$

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$(p,p)$ Poincaré inequality, $\qquad 2 \leq n,\ 1 \leq p < n.$

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Higher order Poincaré inequality with polynomials, $m \in \mathbb{N}$

$$\frac{1}{|Q|}\int_Q |f(y) - \pi_Q(y)|dy \lesssim \frac{\ell(Q)^m}{|Q|}\int_Q |\nabla^m f|\, dy$$

Poincaré-Sobolev inequality

$$\left(\frac{1}{|Q|}\int_Q |f - f_Q|^{p^*} dx\right)^{\frac{1}{p^*}} \lesssim \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p\right)^{\frac{1}{p}}$$

# Poincaré - Sobolev - Classical results

Poincaré-Sobolev inequality

$$\left(\frac{1}{|Q|}\int_Q |f - f_Q|^{p^*}\,dx\right)^{\frac{1}{p^*}} \lesssim \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p\right)^{\frac{1}{p}}$$

$$p^* = \frac{np}{n-p}$$

# Weights

$$[w]_{A_p} := \sup_Q \left( \fint_Q w \right) \left( \fint_Q w^{1-p'} \right)^{p-1}$$

$$\frac{|E|}{|Q|} \leq [w]_{A_p}^{\frac{1}{p}} \left( \frac{w(E)}{w(Q)} \right)^{\frac{1}{p}} \qquad , \qquad E \subset Q$$

$$[w]_{A_1} := \sup_Q \left( \fint_Q w \right) \|w^{-1}\|_{L^\infty(Q)} \qquad , \qquad Mw(x) \leq Cw(x) \text{ a.e. } x$$

$$A_\infty := \bigcup_{p \geq 1} A_p \qquad , \qquad [w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)\, dx$$

# Self improving functionals (pre-history)

Starting point

$$\fint_Q |f - f_Q| d\mu \leq a(Q), \qquad a : \mathcal{Q} \to (0, \infty)$$

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$$\fint_Q |f - f_Q| d\mu \leq a(Q), \qquad a : \mathcal{Q} \to (0, \infty)$$

Hypothesis on the functional $a$

$$\sum_{P \in \Lambda} a(P)^p w(P) \leq C^p a(Q)^p w(Q)$$

$$a \in D_p(w)$$

# Self improving functionals (pre-history)

**Theorem (Franchi-Perez-Wheeden - 1998)**

*Let* $w \in A_\infty$ *and* $a \in D_p(w)$ *for some* $p > 0$. *Let* $f$ *such that*

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq a(Q).$$

*Then*

$$\|f - f_Q\|_{L^{p,\infty}_{Q,\frac{w}{w(Q)}}} \leq C\|a\|a(Q).$$

- Only for the weak norm
- $C$ depends exponentially on $[w]_{A_\infty}$.

# Truncation

Truncation or *weak implies strong* lemma:

**Lemma**

*Let* $g \geq 0$*, Lipschitz. Suppose a weak* $(q, p)$*-type estimate for the measures* $\mu, \nu$ *and* $1 < q \leq p$*:*

$$\sup_{t>0} t\,\mu(\{x \in \mathbb{R}^n : g(x) > t\})^{1/p} \lesssim \left( \int_{\mathbb{R}^n} |\nabla g(x)|^q d\nu \right)^{\frac{1}{q}}$$

*Then the strong estimate also holds, namely*

$$\|g\|_{L^p_\mu} \lesssim \left( \int_{\mathbb{R}^n} |\nabla g(x)|^q d\nu \right)^{\frac{1}{q}}$$

# $D_p$ and Poincaré

**Model example: $p > 0$**

$$a(Q) = [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left( \frac{1}{w(Q)} \int_Q |\nabla f|^p w \right)^{\frac{1}{p}} \quad \rightsquigarrow \quad \begin{cases} a \in D_p(w) \\ \|a\| = 1 \end{cases}$$

**Theorem**

*Let $w \in A_p$, then*

$$\left( \frac{1}{w(Q)} \int_Q |f - f_Q|^p w \right)^{\frac{1}{p}} \leq C [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left( \frac{1}{w(Q)} \int_Q |\nabla f|^p w \right)^{\frac{1}{p}}$$

How to deal with higher order Poincaré inequality with polynomials? No truncation...

$$\frac{1}{|Q|} \int_Q |f(y) - \pi_Q(y)| dy \lesssim \frac{\ell(Q)^m}{|Q|} \int_Q |\nabla^m f| \, dy$$

# New $D_p$-type condition

**Smallness preserving functionals**

$a \in SD_p^s(w)$ for $0 \leq p < \infty$ and $s > 1$ if

$$\sum_i a(Q_i)^p w(Q_i) \leq \|a\|^p \left( \frac{\sum_i |Q_i|}{|Q|} \right)^{\frac{p}{s}} a(Q)^p w(Q)$$

for every collection $\{Q_i\}$ of pairwise disjoint subcubes of $Q$.

# Main Theorem

**Theorem (A)**

*Let* $w \in A_\infty$, $p \geq 1, s > 1$ *and* $a \in SD_p^s(w)$*. If*

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq a(Q),$$

*then*

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^p\, w\right)^{\frac{1}{p}} \leq C_n\, s\|a\|^s a(Q)$$

# About the proof

Hypothesis: $$\fint_Q \frac{|f - f_Q|}{a(Q)} \leq 1, \qquad a \in SD_p^s(w)$$

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Calderon - Zygmund decomposition for $L>1$:

$$\Omega_L := \left\{x \in Q : M_Q^d\left(\frac{|f-f_Q|}{a(Q)}\chi_Q\right)(x) > L\right\} = \bigcup_j Q_j$$

$$L < \fint_{Q_j} \frac{|f-f_Q|}{a(Q)} \leq L\,2^n \qquad , \qquad |\Omega_L| = \left|\bigcup Q_j\right| \leq \frac{|Q|}{L}$$

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**Key step:** Go from $(\cdot)_Q$ to $(\cdot)_{Q_j}$

# About the proof

Calderón - Zygmund decomposition into good and bad parts

$$\frac{f-f_Q}{a(Q)} = g_Q + b_Q, \qquad \begin{cases} |g(x)| \leq 2^n L \\ b_Q(x) = \sum_j \frac{f(x)-f_{Q_j}}{a(Q)} \chi_{Q_j}(x) \end{cases}$$

$$\left( \frac{1}{w(Q)} \int_Q \frac{|f-f_Q|^p}{a(Q)^p} \, wdx \right)^{\frac{1}{p}} \leq 2^n L + \left( \frac{1}{w(Q)} \int_{\Omega_L} \Big| \sum_j b_{Q_j} \Big|^p \, wdx \right)^{\frac{1}{p}}$$

# About the proof

$$
\begin{aligned}
\int_{\Omega_L} |\sum_j b_{Q_j}|^p \, w dx &\leq \sum_i \int_{Q_i} \left| b_{Q_j} \right|^p \, w dx \\
&= \frac{1}{a(Q)^p} \sum_j \frac{a(Q_j)^p w(Q_j)}{w(Q_j)} \int_{Q_j} \left| \frac{f - f_{Q_j}}{a(Q_j)} \right|^p \, w dx \\
&\leq X^p \frac{\sum_j a(Q_i)^p w(Q_j)}{a(Q)^p}
\end{aligned}
$$

where $X$ is the quantity defined by

$$
X = \sup_Q \left( \frac{1}{w(Q)} \int_Q \left| \frac{f - f_Q}{a(Q)} \right|^p \, w dx \right)^{1/p} .
$$

# About the proof

$$\left(\frac{1}{w(Q)}\int_Q \frac{|f-f_Q|^p}{a(Q)^p}\,wdx\right)^{\frac{1}{p}} \leq 2^nL + \left(X^p\,\frac{\sum_i a(Q_i)^p w(Q_i)}{a(Q)^p w(Q)}\right)^{\frac{1}{p}}$$

$$X \leq 2^nL + X\frac{\|a\|}{L^{1/s}}.$$

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Choose $L = 2e\,\text{máx}\{\|a\|^s, 1\}$ to conclude that

$$X \leq 2^n 2e\|a\|^s\left((2e)^{1/s}\right)' \leq e2^{n+1}s\|a\|^s$$

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! Is $X$ finite?

# About the proof

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!! Where is the $A_\infty$ condition? Why there is no $[w]_{A_\infty}$?

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[LLO] Lerner, Lorist, Ombrosi proved that $A_\infty$ is not required (*)

# Consequences I: weighted $(p, p)$ Poincaré

**Model example: $\alpha, p > 0$**

$$a(Q) = \ell(Q)^{\alpha}\left(\frac{1}{w(Q)}\mu(Q)\right)^{1/p} \quad \rightsquigarrow \quad \begin{cases} a \in SD_p^{n/\alpha}(w) \\ \|a\| = 1 \end{cases}$$

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From unweighted $(1,1)$ to weighted $(p,p)$ Poincaré inequalities

$$\fint_Q |f - f_Q| dx \;\lesssim\; \ell(Q) \fint_Q |\nabla f| dx$$

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From unweighted $(1,1)$ to weighted $(p,p)$ Poincaré inequalities

$$\begin{aligned} \fint_Q |f - f_Q| dx &\lesssim \ell(Q) \fint_Q |\nabla f| dx \\ &\lesssim [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w \, dx\right)^{\frac{1}{p}} \end{aligned}$$

We have then the starting point:

$$\fint_Q |f - f_Q| dx \lesssim a(Q) \in SD_p^n(w)$$

# Consequences I: weighted $(p,p)$ Poincaré

**Corollary (Theorem A - $A_p$ case)**

*Let $w \in A_p$, $p \geq 1, n > 1$. Since*

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left( \frac{1}{w(Q)} \int_Q |\nabla f|^p w \, dx \right)^{\frac{1}{p}} =: a(Q)$$

*and $a \in SD_p^s(w)$ with $s = n$, $\|a\| = 1$, then*

$$\left( \frac{1}{w(Q)} \int_Q |f - f_Q|^p \, w \right)^{\frac{1}{p}} \leq C_n [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left( \frac{1}{w(Q)} \int_Q |\nabla f|^p w \right)^{\frac{1}{p}}$$

# Consequences II: weighted $(p_w^*, p)$ Poincaré - Sobolev

Unweighted Poincaré-Sobolev

$$\left(\fint_Q |f - f_Q|^{p^*} dx\right)^{\frac{1}{p^*}} \lesssim \ell(Q)\left(\fint_Q |\nabla f|^p\right)^{\frac{1}{p}}, \qquad \frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}$$

Again, we start from:

$$\fint_Q |f - f_Q| dx \lesssim [w]_{A_p}^{\frac{1}{p}} \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\, dx\right)^{\frac{1}{p}} = a(Q)$$

**Goal**

Obtain that $a \in SD^s_{p^*_w}$ with some control on $p^*_w, s$ and $\|a\|$.

# Consequences II: weighted $(p_w^*, p)$ Poincaré - Sobolev

Back to the model example:

$$a(Q) = \ell(Q) \left( \frac{\mu(Q)}{w(Q)} \right)^{1/p}$$

Modified Poincaré-Sobolev index: $p^* = p^*(q, M), \quad (q \geq 1, M > 1)$

$$\frac{1}{p} - \frac{1}{p^*} = \frac{1}{nqM}$$

**Lemma**

$$w \in A_q, 1 \leq q \leq p \Longrightarrow a \in SD_{p^*}^s(w),\ \ s = nM',\ \ \|a\| = [w]_{A_q}^{\frac{1}{nqM}}$$

Key property:

$$\left( \frac{|E|}{|Q|} \right)^q \leq [w]_{A_q} \frac{w(E)}{w(Q)}$$

# Consequences II: weighted $(p_w^*, p)$ Poincaré - Sobolev

Choosing $M = 1 + \frac{1}{q}\log[w]_{A_q}$, we obtain

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n(q + \log[w]_{A_q})}$$

**Theorem (B)**

*Let $1 \leq p < n$ and let $w \in A_q$ be a* ***nontrivial*** *weight with $1 \leq q \leq p$. If*

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq a(Q) = \ell(Q)\left(\frac{1}{w(Q)}\mu(Q)\right)^{1/p},$$

*then*

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^{p_w^*}\, w\right)^{\frac{1}{p_w^*}} \leq C_n \frac{1 + \log[w]_{A_q}^{\frac{1}{q}}}{\log[w]_{A_q}^{\frac{1}{q}}}\, a(Q).$$

# Further improvements on $p_w^*$ for flat weights

$$\frac{1}{p}-\frac{1}{p_w^*}=\frac{1}{n(q+\log[w]_{A_q})}$$

**Theorem (C)**

*Let $1\leq p<n$ and let $w\in A_q$ be a* ***any*** *weight with $1\leq q\leq p$. If*

$$\frac{1}{|Q|}\int_Q|f-f_Q|\leq a(Q)=\ell(Q)\left(\frac{1}{w(Q)}\mu(Q)\right)^{1/p},$$

*then*

$$\|f-f_Qf\|_{L^{p_w^*,\infty}\left(Q,\frac{wdx}{w(Q)}\right)}\leq c_n a(Q)$$

Only for the weak norm...

# Further improvements on $p_w^*$ for flat weights

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n(q + \log[w]_{A_q})}$$

**Corollary**

*Let $1 \leq p < n$ and let $w \in A_q$ be a* ***any*** *weight with $1 \leq q \leq p$.*

$$\left(\frac{1}{w(Q)} \int_Q |f - f_Q|^{p_w^*} w\right)^{\frac{1}{p_w^*}} \leq C_n [w]_{A_p}^{\frac{1}{p}} \left(\frac{\ell(Q)^p}{w(Q)} \int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

# $A_1$ weights, which one is better?

For $A_1$ weights, we can define $p_w^*$ as

$$\frac{1}{p}-\frac{1}{p_p^*}=\frac{1}{n(p+\log[w]_{A_p})} \quad \text{or} \quad \frac{1}{p}-\frac{1}{p_1^*}=\frac{1}{n(1+\log[w]_{A_1})}$$

Compare

$$\frac{1}{n(p+\log[w]_{A_p})}\leq\frac{1}{n(1+\log[w]_{A_1})}$$

Equivalently,

$$[w]_{A_1}\leq e^{p-1}[w]_{A_p}$$

Is this true? Always? Never? Sometimes?

# Further improvements on $p_w^*$ - For the gradient

**Theorem (D - Old stuff works)**

*Let $1 \leq p < n$, $w \in A_q$ with $1 \leq q \leq p$. Let $\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{nq}$. Then*

$$\left( \frac{1}{w(Q)} \int_Q |f - f_Q|^{p_w^*} w \right)^{\frac{1}{p_w^*}} \lesssim [w]_{A_q}^{\frac{1}{nq}} [w]_{A_p}^{\frac{2}{p}} \left( \frac{\ell(Q)^p}{w(Q)} \int_Q |\nabla f|^p w \right)^{\frac{1}{p}}$$

Starting point:

$$⨍_Q |f - f_Q| dx \lesssim [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left( \frac{1}{w(Q)} \int_Q |\nabla f|^p w \, dx \right)^{\frac{1}{p}} = a(Q)$$

Smallness preservation:

- $a \in SD_p^n$ with $\|a\| = 1$
- $w \in A_q \implies a \in D_{p_w^*}(w)$ with $\|a\| = [w]_{A_q}^{\frac{1}{nq}}$

# The case of $A_1$ weights

From Theorem (D): $\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}$, (Sobolev!)

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^{p^*} w\right)^{\frac{1}{p^*}} \lesssim [w]_{A_1}^{\frac{1}{n}} [w]_{A_p}^{\frac{2}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

# Extensions and variations

# Product spaces I: arbitrary rectangles

$\mathcal{R}$: rectangles in $\mathbb{R}^n$ seen as $n$-fold product of intervals on $\mathbb{R}$

Pretty much the same theory, considering the eccentricity of a rectangle

$$e(R) := \frac{|R|^{\frac{1}{n}}}{d(R)}$$

to handle the absence of the formula $|Q| = \ell(Q)^n$

# Product spaces II: product of cubes

$\mathfrak{R}$: $R = I_1 \times I_2 \subset \mathbb{R}^n$ where $I_1 \subset \mathbb{R}^{n_1}$ and $I_2 \subset \mathbb{R}^{n_2}$ are cubes.

Starting point (requires a proof):

$$\fint_R |f - f_R| \lesssim \ell(I_1) \fint_R |\nabla_1 f| + \ell(I_2) \fint_R |\nabla_2 f|$$

Conclusion (by self-improving):

$$\|f - f_R\|_{L^{p^*}(\mu)} \leq C[w]_{A_{q,\mathfrak{R}}}^{\frac{1}{p}} \left[\ell(I_1) \|\nabla_1 f\|_{L^p(\mu)} + \ell(I_2) \|\nabla_2 f\|_{L^p(\mu)}\right],$$

where

$$\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n} \frac{1}{q + \log[w]_{A_{q,\mathfrak{R}}}} \qquad \text{and} \qquad \mu = \frac{wdx}{w(R)}$$

The geometric information related to the eccentricity here is

$$E(R) = \frac{\ell(I_2)^{n_2}}{\ell(I_1)^{n_2}} \qquad , \qquad \ell(I_1)^n \cdot E(R) = |R|.$$

# Minimal conditions for BMO

Let us start by recalling that a function $f$ belongs to BMO if

$$\|f\|_{\mathrm{BMO}} := \sup_Q \fint_Q |f - f_Q| < \infty$$

We also recall that

$$\|f\|_{\mathrm{BMO}} \simeq \sup_Q \inf_{c\in\mathbb{R}} \fint_Q |f - c|$$

$$\|f\|_{\varphi,Q} := \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \varphi\left(\frac{|f|}{\lambda}\right) dx \leq 1\right\}$$

$$\|f\|_{\mathrm{BMO}_\varphi} := \sup_Q \inf_{c\in\mathbb{R}} \|f - c\|_{\varphi,Q} < \infty.$$

**Theorem (A)**

*Let $\varphi$ be an increasing,* ***concave*** *function with $\varphi(0)=0$ and such that $\lim_{t\to\infty}\varphi(t)=+\infty$. Then $\mathrm{BMO}_\varphi=\mathrm{BMO}$ with the following quantitative estimates:*

$$\varphi^{-1}(1)\|f\|_{\mathrm{BMO}_\varphi}\leq\|f\|_{\mathrm{BMO}}$$

*and*

$$\|f\|_{\mathrm{BMO}}\leq\left(2\varphi^{-1}(4)+\varphi^{-1}\left(2+2^{n+2}\right)\right)\|f\|_{\mathrm{BMO}_\varphi}$$

Logunov, Slavin, Stolyarov, Vasyunin, Zatitskiy [LSSVZ15]

$$K_{\varphi,Q}(f)=\sup_{J\text{ subcube of }Q}\fint_J\varphi\left(|f-f_J|\right).$$

# Minimal conditions for BMO - Sketch of proof

Hypothesis:

$$\|f\|_{\mathrm{BMO}_\varphi} = 1 \qquad , \qquad \fint_Q \varphi\left(|f - c_Q|\right) \leq 2$$

Goal: to control

$$X = \sup_{Q \text{ cube}} \fint_Q |f(x) - c_Q|\, dx$$

Calderón-Zygmund:

- $L < \fint_{Q_j} \varphi\left(|f - c_Q|\right) \leq 2^n L,$
- $\varphi\left(|f(x) - c_Q|\right) \leq L \quad$ a.e. $x \in Q \setminus \bigcup_j Q_j,$
- $\frac{1}{|Q|} \sum_j |Q_j| \leq \frac{2}{L}$

# Minimal conditions for BMO - Sketch of proof

Standard first step:

$$\left|f(x) - c_Q\right| \leq \left|f(x) - c_{Q_j}\right| + \left|c_Q - c_{Q_j}\right|$$

Key step:

$$\begin{aligned}
\left|c_Q - c_{Q_j}\right| &= \varphi^{-1}\left(\fint_{Q_j} \varphi\left(\left|c_Q - c_{Q_j}\right|\right)\right) \\
&\leq \varphi^{-1}\left(\fint_{Q_j} \varphi\left(\left|f(x) - c_Q\right|\right) dx + \fint_{Q_j} \varphi\left(\left|f(x) - c_{Q_j}\right|\right) dx\right) \\
&\leq \varphi^{-1}\left(2^n L + 2\right) \qquad\qquad \text{(subaditivity)}
\end{aligned}$$

# Minimal conditions for BMO - Sketch of proof

Self absorbing:

$$\begin{aligned} \fint_Q |f(x) - c_Q| &\leq \frac{1}{|Q|}\int_{Q\setminus\bigcup_j Q_j} |f(x) - c_Q| + \frac{1}{|Q|}\int_{\bigcup_j Q_j} |f(x) - c_Q| \\ &\leq \varphi^{-1}(L) + \sum_j \frac{|Q_j|}{|Q|}\fint_{Q_j} \left|f(x) - c_{Q_j}\right| + \left|c_Q - c_{Q_j}\right| \\ \mathbf{X} &\leq \varphi^{-1}(L) + \frac{2\mathbf{X}}{L} + \frac{2\varphi^{-1}(2^n L + 2)}{L} \end{aligned}$$

Simply choose $L = 4$ to get

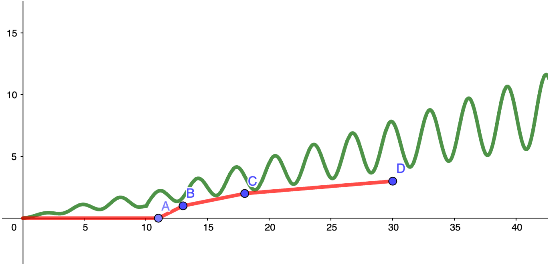
$$\mathbf{X} \leq 2\varphi^{-1}(4) + \varphi^{-1}(2 + 2^{n+2})$$

# Minimal conditions for BMO - Extensions

**Theorem**

*Let* $\psi : [0, \infty) \to [0, \infty)$ *be* ***any measurable*** *function such that* $\psi(0) = 0$ *and* $\lim_{t\to\infty} \psi(t) = +\infty$. *Then*

$$\|f\|_{\mathrm{BMO}} \leq c_{n,\psi} \|f\|_{\mathrm{BMO}_\psi}.$$



# Minimal conditions for BMO

**Spaces of Homogeneous Type:** $(X, d, \mu)$.

Vitali covering lemma $\rightsquigarrow$ C-Z decomposition

**Non doubling measures (non atomic):** $\mu(\partial Q) = 0$

Besicovitch covering lemma $\rightsquigarrow$ Besicovitch-C-Z decomposition

**Rectangles and non-doubling measures (non atomic):** $\mu(\partial R) = 0$

Riesz's Rising Sun lemma $\rightsquigarrow$ C-Z decomposition

# Summary and a Template

- A reasonable inequality as a starting point

$$\fint_Q |f - f_Q| dx \lesssim a(Q).$$

- A functional $a(Q)$ with some chance of satisfying a smallness preservation property.

- A geometric structure compatible with Calderón-Zygmund decompositions.

# Work in progress

- Trace-Poincaré-Sobolev (with Andrea Olivo (BCAM))

$$\frac{1}{\mathcal{H}^{n-1}(\partial Q)} \int_{\partial Q} |f - f_{\partial Q}| d\mathcal{H}^{n-1} \lesssim d(Q) \frac{1}{|Q|} \int_{Q} |\nabla f| dx$$

- Selfimproving on nested fractals (Felipe Negreira (UBA))

- Optimality of the exponents $p_w^*$ (Alejandro Claros (BCAM))

Thank you!

Thank you!

# References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# Some extra stuff

# Higher order Poincaré

## Theorem

*Let $w$ be any weight, $p \geq 1$, $a \in SD_p^s(w)$*
*If*

$$\frac{1}{|Q|}\int_Q |f - P_Q f| \leq a(Q),$$

*then*

$$\left(\frac{1}{w(Q)}\int_Q |f - P_Q f|^p \, w dx\right)^{\frac{1}{p}} \leq C_{n,m}(1+s)\max\{\|a\|^s, 1\} a(Q)$$

## Corollary

*Let $1 \leq p < \frac{n}{m}$ and let $w \in A_p$.*

$$\left(\frac{1}{w(Q)}\int_Q |f - P_Q f|^p w\right)^{\frac{1}{p}} \lesssim [w]_{A_p}^{\frac{1}{p}} \ell(Q)^m \left(\frac{1}{w(Q)}\int_Q |\nabla^m f|^p w\right)^{\frac{1}{p}}$$

# The model example and $SD_p^{n/\alpha}$

Model example: $\alpha, p > 0$

$$a(Q) = \ell(Q)^\alpha \left(\frac{1}{w(Q)}\mu(Q)\right)^{1/p} \quad \rightsquigarrow \quad \begin{cases} a \in SD_p^{n/\alpha}(w) \\ \|a\| = 1 \end{cases}$$

$$\begin{aligned}
\sum_i a(Q_i)^p w(Q_i) &\leq \sum_i \ell(Q_i)^{p\alpha}\mu(Q_i) = \sum_i |Q_i|^{\frac{p\alpha}{n}}\mu(Q_i) \\
\left(\frac{p\alpha}{n} < 1\right) \qquad &\leq \left(\sum_i |Q_i|\right)^{\frac{p\alpha}{n}} \left(\sum_i \mu(Q_i)^{(\frac{n}{p\alpha})'}\right)^{\frac{1}{(\frac{n}{p\alpha})'}} \\
&\leq \left(\frac{|Q|}{L}\right)^{\frac{p\alpha}{n}} \sum_i \mu(Q_i) \\
&= \left(\frac{1}{L}\right)^{\frac{p\alpha}{n}} \ell(Q)^{p\alpha}\mu(Q) = \left(\frac{1}{L}\right)^{\frac{p\alpha}{n}} a(Q)^p w(Q).
\end{aligned}$$

# Consequences II: weighted $(p^*_w, p)$ Poincaré - Sobolev

$$
\begin{aligned}
\sum_i a(Q_i)^{p^*} w(Q_i) &= \sum_i \mu(Q_i)^{\frac{p^*}{p}} \left( \frac{\ell(Q_i)}{w(Q_i)^{\frac{1}{p}-\frac{1}{p^*}}} \right)^{p^*} \\
&= \sum_i \mu(Q_i)^{\frac{p^*}{p}} \left( \frac{|Q_i|}{w(Q_i)^{\frac{1}{qM}}} \right)^{\frac{p^*}{n}} \\
&\leq [w]_{A_q}^{\frac{p^*}{nqM}} \left( \frac{|Q|^q}{w(Q)} \right)^{\frac{p^*}{nqM}} \sum_i \mu(Q_i)^{\frac{p^*}{p}} |Q_i|^{\frac{p^*}{nM'}}
\end{aligned}
$$

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$$
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\end{aligned}
$$

Hölder's inequality plus some magic...

# Consequences II: weighted $(p_w^*, p)$ Poincaré - Sobolev

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\sum_i a(Q_i)^{p^*} w(Q_i) &= \sum_i \mu(Q_i)^{\frac{p^*}{p}} \left( \frac{\ell(Q_i)}{w(Q_i)^{\frac{1}{p}-\frac{1}{p^*}}} \right)^{p^*} \\
&= \sum_i \mu(Q_i)^{\frac{p^*}{p}} \left( \frac{|Q_i|}{w(Q_i)^{\frac{1}{qM}}} \right)^{\frac{p^*}{n}} \\
&\leq [w]_{A_q}^{\frac{p^*}{nqM}} \left( \frac{|Q|^q}{w(Q)} \right)^{\frac{p^*}{nqM}} \sum_i \mu(Q_i)^{\frac{p^*}{p}} |Q_i|^{\frac{p^*}{nM'}}
\end{aligned}
$$

Hölder's inequality plus some magic...

$$
\leq [w]_{A_q}^{\frac{p^*}{nqM}} a(Q)^{p^*} w(Q) \left( \frac{1}{L} \right)^{\frac{p^*}{nM'}}
$$

# Measure-dyadic decompositions

$I \subset \mathbb{R}$:

- $G_1(I) = \{I_+, I_-\}$ such that $\mu(I_+) = \mu(I_-) = \mu(I)/2$.
- $G_2(I) = G_1(I_+) \cup G_1(I_-)$.
- Recursively define $G_n$ to obtain $\mathcal{D}_I^\mu$

Chains

$$\mathcal{C} = \{J_i\}_{i \in \mathbb{N}}, \qquad J_i \in G_i(I), \qquad J_{i+1} \subset J_i$$

$$\mathcal{C}_\infty := \bigcap_{J \in \mathcal{C}} J \qquad \mu(\mathcal{C}_\infty) = 0$$

$$E := I \setminus \bigcup_{|\mathcal{C}_\infty| > 0} \mathcal{C}_\infty, \qquad \mu(I) = \mu(E)$$

# Hardy-Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)| \, dy,$$

$$M : L^p(w\,dx) \to L^p(w\,dx) \iff w \in A_p \qquad 1 < p < \infty$$

$$M : L^1(w\,dx) \to L^{1,\infty}(w\,dx) \iff w \in A_1$$

$$\|M\|_{L^p(w)} \lesssim p'[w]_{A_p}^{\frac{1}{p-1}}, \qquad 1 < p < \infty$$

$$\|M\|_{L^{p,\infty}(w)} \approx [w]_{A_p}^{\frac{1}{p}}, \qquad 1 \leq p < \infty$$