On an Intriguing Inequality

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Prove or Disprove

For any $n \in \mathbb{N}$, and any selection of unit vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, there exists a unit vector $x \in \mathbb{R}^n$ such that

$$\langle x, v_1 \rangle \langle x, v_2 \rangle \cdots \langle x, v_n \rangle \geq \frac{1}{\sqrt{n^n}}.$$

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Dangling the carrot: the previous statement is true if we replace $\mathbb{R}^n$ with $\mathbb{C}^n$.

Introduction

Given a Banach space E, a mapping $P : E \to \mathbb{K}$ is called a continuous m-homogeneous polynomial if there exists a continuous m-linear form $\Phi \in \mathscr{L}(^m E; \mathbb{K})$ such that

$$P(x) = \Phi(\underbrace{x, \ldots, x}_{m\text{-times}}) \text{ for all } x \in E.$$

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Let us denote by $\mathscr{P}(^m E)$ the space of continuous m-homogeneous polynomials defined on E, and consider the norm $\|P\|_{\mathscr{P}}$ on $\mathscr{P}(^m E)$, defined as follows:

$$\|P\|_{\mathscr{P}} = \sup_{\|x\|_E = 1} |P(x)|,$$

which corresponds to the uniform norm over the unit sphere of E.

Introduction

For a Banach space E with dual space E', C. Benítez, Y. Sarantopoulos and A. Tonge ('98) defined the $n-$th linear polarization constant of E :

$$\mathrm{c}_n(E) = \inf\{M > 0 : \|\phi_1\|_{\mathscr{P}} \cdots \|\phi_n\|_{\mathscr{P}} \leq M \|\phi_1 \cdots \phi_n\|_{\mathscr{P}}, \forall\, \phi_1, \ldots \phi_n \in E'\},$$

where where $\phi_1 \cdots \phi_n$ is the $n-$homogeneous polynomial defined as the pointwise product

$$\phi_1 \cdots \phi_n(x) = \phi_1(x) \cdots \phi_n(x).$$

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Namely, given an infinite dimensional Hilbert space $\mathbb{H}$, we have $\mathrm{c}_n(\mathbb{H}) \leq \mathrm{c}_n(E)$ for any infinite dimensional Banach space E.

Motivation

In a Hilbert space $\mathbb{H}$, the Riesz representation theorem allows us to express the inequality

$$\|\phi_1\|_{\mathscr{P}} \cdots \|\phi_n\|_{\mathscr{P}} \leq \mathsf{c}_n(\mathbb{H}) \, \|\phi_1 \cdots \phi_n\|_{\mathscr{P}},$$

in the equivalent form:

$$\|v_1\|_{\mathbb{H}} \cdots \|v_n\|_{\mathbb{H}} \leq \mathsf{c}_n(\mathbb{H}) \sup_{\|x\|_{\mathbb{H}}=1} |\langle x, v_1\rangle \cdots \langle x, v_n\rangle|.$$

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By a density argument we may assume that the set of linear functionals, and so de set of vectors $\{v_1, \ldots, v_n\}$, is linearly independent. Since $\mathrm{span}\{v_1, \ldots, v_n\}$ is isometrically isomorphic to ℓ_2^n, it follows that $\mathsf{c}_n(\mathbb{H}) = \mathsf{c}_n(\ell_2^n)$. Hence, finding the exact value of $\mathsf{c}_n(\ell_2^n)$ is an interesting and natural problem.

The Problem: $c_n(\ell_2^n) = \sqrt{n^n}$?

In this talk, we focus on the ℓ_2^n case: given any set of vectors $\{v_i\}_{i=1}^n \subset \ell_2^n$, we study the inequality

$$\|v_1\|_2 \cdots \|v_n\|_2 \leq c_n(\ell_2^n) \sup_{\|x\|_2=1} |\langle x, v_1\rangle \cdots \langle x, v_n\rangle|.$$

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Given an orthonormal basis $\{e_i\}_{i=1}^n \subset \ell_2^n$, the Arithmetic-Geometric Mean inequality implies that, for any unit vector $x \in \ell_2^n$, we have

$$\prod_{i=1}^n |\langle x, e_i\rangle| = \left(\prod_{i=1}^n |\langle x, e_i\rangle|^2\right)^{1/2} \leq \left(\frac{1}{n}\sum_{i=1}^n |\langle x, e_i\rangle|^2\right)^{n/2} = \frac{1}{\sqrt{n^n}}.$$

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From this bound, it follows that $c_n(\ell_2^n) \geq \sqrt{n^n}$.

The complex case

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Then we have $\mathrm{c}_n(\mathbb{C}^n) = n^{n/2}$.

The real case - Conjecture

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Conjecture

Given n unit vectors $\{v_i\}_{i=1}^n \subset \mathbb{R}^n$, then

$$\sup_{\|x\|_2=1} |\langle x, v_1\rangle \cdots \langle x, v_n\rangle| \geq \frac{1}{\sqrt{n^n}},$$

and equality holds if and only if $\{v_i\}_{i=1}^n$ forms an orthonormal system.

Back to the complex case...

Note that even in the complex case, the proofs given by J. Arias-de-Reyna and K. Ball did not show if the equality

$$\sup_{\|z\|_2=1} |\langle z, v_1\rangle \cdots \langle z, v_n\rangle| = \frac{1}{\sqrt{n^n}}$$

holds only if $\{v_i\}_{i=1}^n$ is an orthonormal system.

Lower Bounds for Products of Polynomials

Theorem

Let $P_i \in \mathscr{P}({}^{m_i}\mathbb{C}^n)$ for $i = 1, \ldots, n$. Then, the following inequality holds:

$$\|P_1\|_{\mathscr{P}} \cdots \|P_n\|_{\mathscr{P}} \leq \sqrt{\frac{(m_1 + \cdots + m_n)^{(m_1+\cdots+m_n)}}{m_1^{m_1} \cdots m_n^{m_n}}} \, \|P_1 \cdots P_n\|_{\mathscr{P}}.$$

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In the case of linear functionals, that is, when $P_i \in \mathscr{P}({}^1\mathbb{C}^n)$ for $i = 1, \dots, n$, the previous theorem asserts that

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$$\|P_1\|_{\mathscr{P}} \cdots \|P_n\|_{\mathscr{P}} \leq \sqrt{n^n} \|P_1 \cdots P_n\|_{\mathscr{P}} \quad \Longrightarrow \quad \mathrm{c}_n(\mathbb{C}^n) \leq n^{n/2}.$$

Products of linear functionals - The complex case - Minimum condition

Theorem

Let $\{v_k\}_{k=1}^n$ *be unit vectors in a complex Hilbert space* $\mathbb{H}$ *and* $P_k(z) = \langle z, v_k \rangle$ *for* $1 \leq k \leq n$. *If* $1 = \sqrt{n^n}\, \|P_1 \cdots P_n\|_{\mathscr{P}}$, *then* $\{v_k\}_{k=1}^n$ *is an orthonormal system.*

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Lemma

Let v_1, v_2 be unit vectors in a complex Hilbert space. If $\sup_{\|z\|_2=1} |\langle z, v_1\rangle \langle z, v_2\rangle| = \frac{1}{2}$, *then* $\langle v_1, v_2\rangle = 0$.

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$$1 = \sqrt{n^n}\,\|P_1 \cdots P_n\|_{\mathscr{P}} \ge \sqrt{n^n}\sqrt{\frac{2^2}{n^n}}\,\|P_1 P_2\|_{\mathscr{P}}\,\|P_3\|_{\mathscr{P}} \cdots \|P_n\|_{\mathscr{P}}$$

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&= 2\,\|P_1P_2\|_{\mathscr{P}} \geq 2\sqrt{\frac{1}{2^2}}\,\|P_1\|_{\mathscr{P}}\,\|P_2\|_{\mathscr{P}}
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$$= 2\,\|P_1 P_2\|_{\mathscr{P}} \ge 2\sqrt{\frac{1}{2^2}}\,\|P_1\|_{\mathscr{P}}\,\|P_2\|_{\mathscr{P}} = 1$$

In particular, $\|P_1 P_2\|_{\mathscr{P}} = 1/2$. Therefore $\langle v_1, v_2\rangle = 0$. □

The $n-$th linear polarization constant of $\mathbb{R}^n$.

As mentioned earlier, to prove the equality $c_n(\mathbb{R}^n) = \sqrt{n^n}$, it is enough to prove that for any set of unit vectors $\{v_i\}_{i=1}^n \subset \mathbb{R}^n$, there exists a norm one vector $x \in \mathbb{R}^n$ such that

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The existence of such a vector for $n = 2, 3, 4$ and 5 was shown by A. Pappas and S. G. Révész, based on an appropriate choice of signs $\{\varepsilon_i\}_{i=1}^n$ that maximizes the Euclidean norm of $\sum_{i=1}^n \varepsilon_i v_i$.

An Intriguing Inequality

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For $2 \leq n \leq 14$, and any set of n unit vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, there exists a unit vector $x \in \mathbb{R}^n$ such that

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Corollary

Given $2 \leq n \leq 14,$ *then* $\mathrm{c}_n(\mathbb{R}^n) = \sqrt{n^n}.$

Ideas behind the proof

Let us consider $n \in \mathbb{N}, n \geq 2, s \in [\sqrt{n}, n]$, and the cube $Q_s = [s^{-1}, 1]^n \subset \mathbb{R}^n$.

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Given the function $f : Q_s \to \mathbb{R}$, defined by $f(a) = a_1 \cdots a_n$, we are interested in finding the constrained minima

$$\min_{a \in Q_s} f(a), \text{ subject to } \sum_{i=1}^{n} a_i = s.$$

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We define Σ_s as the set of points in Q_s that satisfy the linear constraint $\sum_{i=1}^{n} a_i = s$, that is

$$\Sigma_s = Q_s \cap \left\{ a \in \mathbb{R}^n : \sum_{i=1}^{n} a_i = s \right\}.$$

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We aim to understand how the minimum value of f behaves under the constraint. To formalize this, we define the function $\mu : [\sqrt{n}, n] \to \mathbb{R}$, by $\mu(s) = \min_{a \in \Sigma_s} f(a)$.

Ideas behind the proof

By the Arithmetic-Geometric mean inequality, we have that

$$\max_{a \in \Sigma_s} f(a) = f\left(\frac{s}{n}, \frac{s}{n}, \ldots, \frac{s}{n}\right) = \left(\frac{s}{n}\right)^n.$$

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Thus, it is natural to suspect that $\mu(s)$ is attained at the intersection of the hyperplane $\sum_{i=1}^{n} a_i = s$ and one of the cube's faces, where some coordinates take the smallest possible value.

Ideas behind the proof

Following this idea, let us define

$$k_0(s) = \min\left\{k \in \mathbb{N} : ks^{-1} + n - k < s\right\}.$$

This value gives us the first coordinate k, such that any point $a \in Q_s = [s^{-1}, 1]^n$,

$$a = (\underbrace{s^{-1}, s^{-1}, \ldots, s^{-1}}_{k_0(s)-times}, a_{k_0(s)+1}, \ldots, a_n)$$

does not belong to Σ_s.

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$$k_0(s) = \min\left\{k \in \mathbb{N} : ks^{-1} + n - k < s\right\}.$$

This value gives us the first coordinate k, such that any point $a \in Q_s = [s^{-1}, 1]^n$,

$$a = (\underbrace{s^{-1}, s^{-1}, \dots, s^{-1}}_{k_0(s)-times}, a_{k_0(s)+1}, \dots, a_n)$$

does not belong to Σ_s.

After a few computations...

$$k_0(s) = \left\lfloor \frac{s(n-s)}{s-1} \right\rfloor + 1$$

Auxiliary results

We can find an explicit formula for $\mu(s)$, namely we have

$$\mu(s) = \frac{s^{-1} + s - n + \left(\left\lfloor \frac{s(n-s)}{s-1} \right\rfloor + 1 \right) (1 - s^{-1})}{s^{\left\lfloor \frac{s(n-s)}{s-1} \right\rfloor}}.$$

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Theorem

Given $n \in \mathbb{N}$, $2 \leq n \leq 14$, let $f : Q_s \to \mathbb{R}$ be the function defined by $f(a_1, a_2, \ldots, a_n) = a_1 a_2 \cdots a_n$. If we consider $\mu : [\sqrt{n}, n] \to \mathbb{R}$, where $\mu(s) = \min_{a \in \Sigma_s} f(a)$, then

$$\mu(s) \geq \frac{1}{\sqrt{n^n}}.$$

Moreover, the minimum is attained only at $s = \sqrt{n}$.

A picture is worth a thousand words

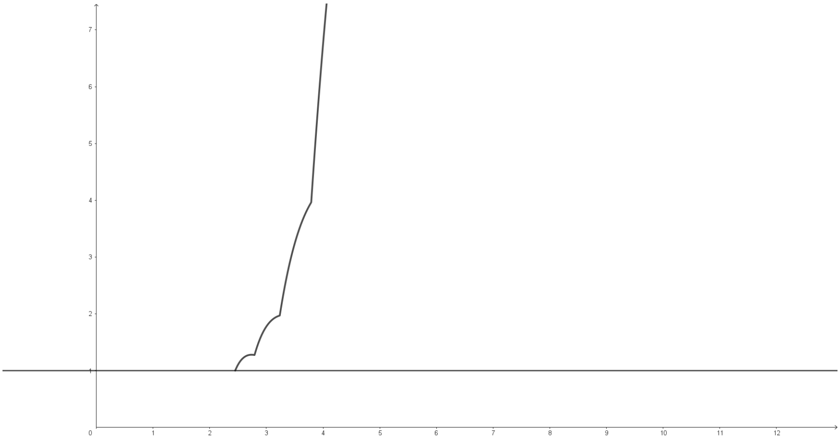

Figure: n=6

A picture is worth a thousand words

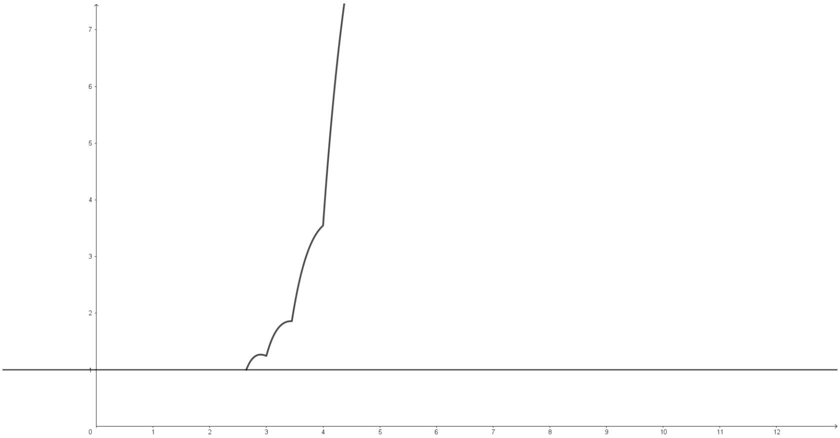

Figure: n=7

A picture is worth a thousand words

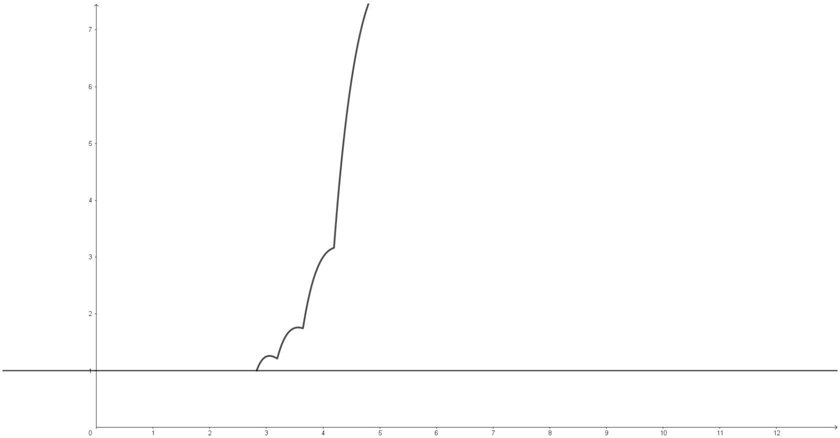

Figure: n=8

A picture is worth a thousand words

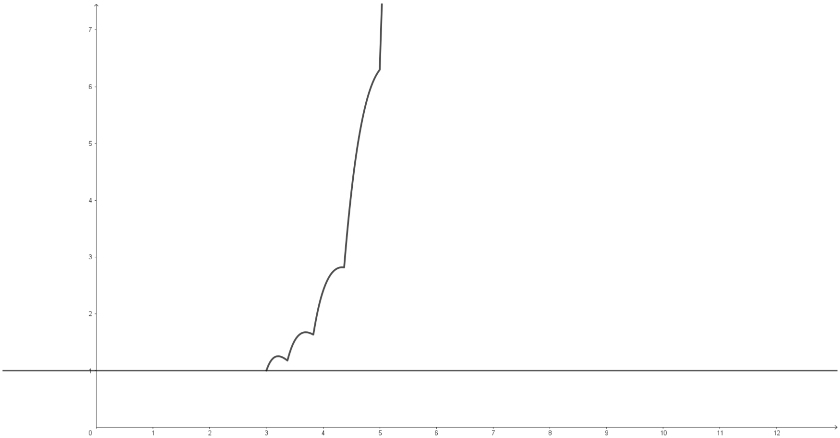

Figure: n=9

A picture is worth a thousand words

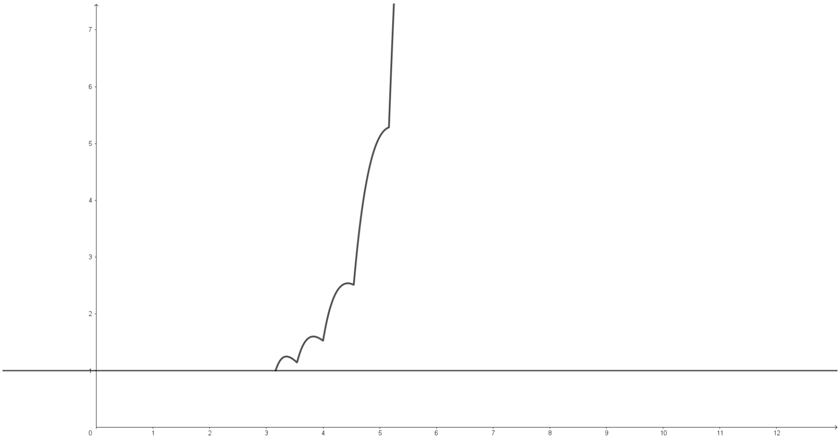

Figure: n=10

A picture is worth a thousand words

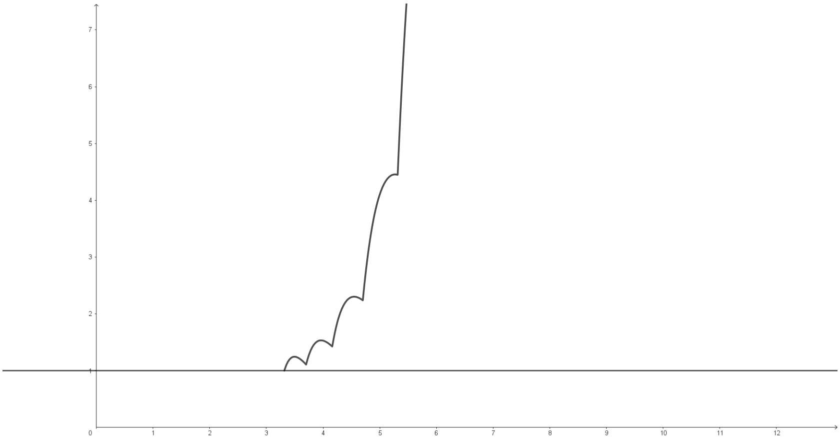

Figure: n=11

A picture is worth a thousand words

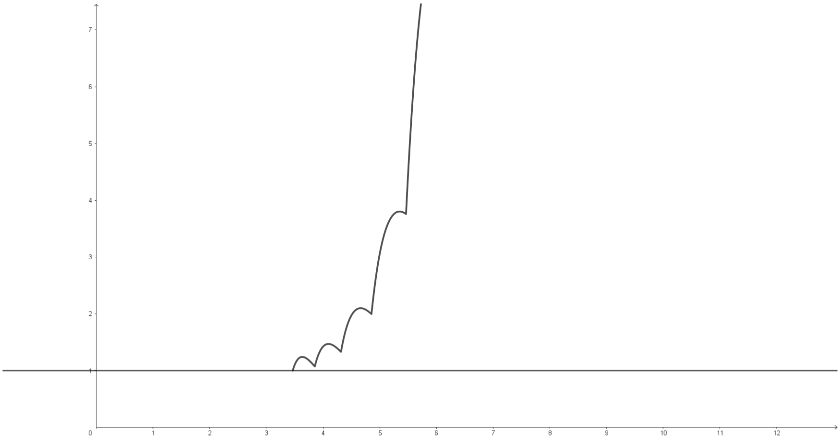

Figure: n=12

A picture is worth a thousand words

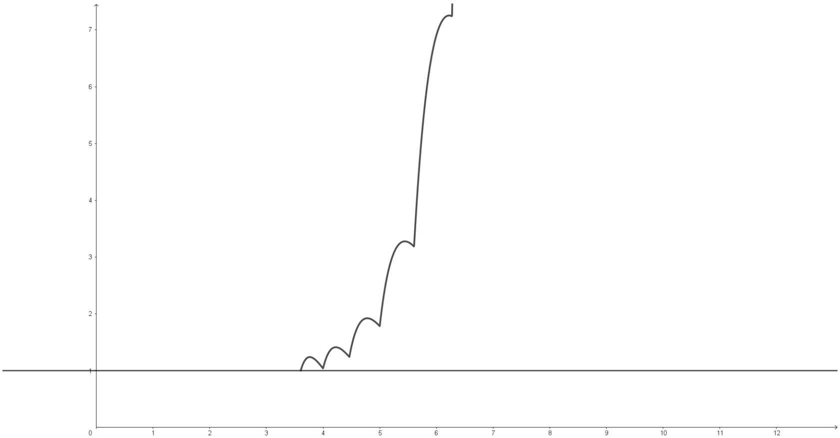

Figure: n=13

A picture is worth a thousand words

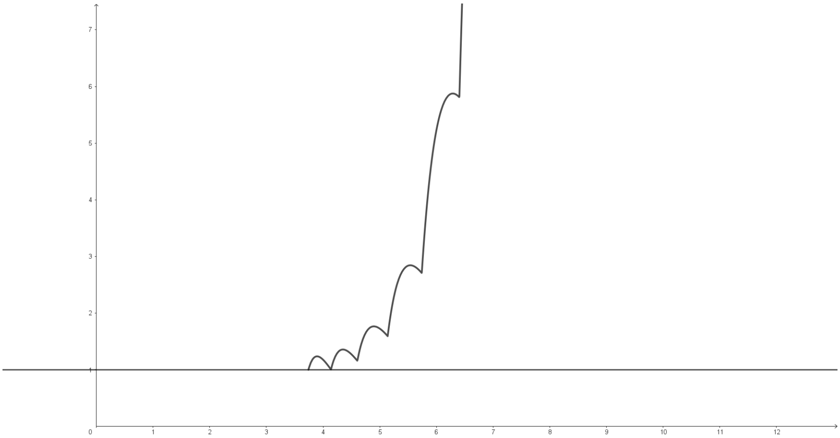

Figure: n=14

A picture is worth a thousand words

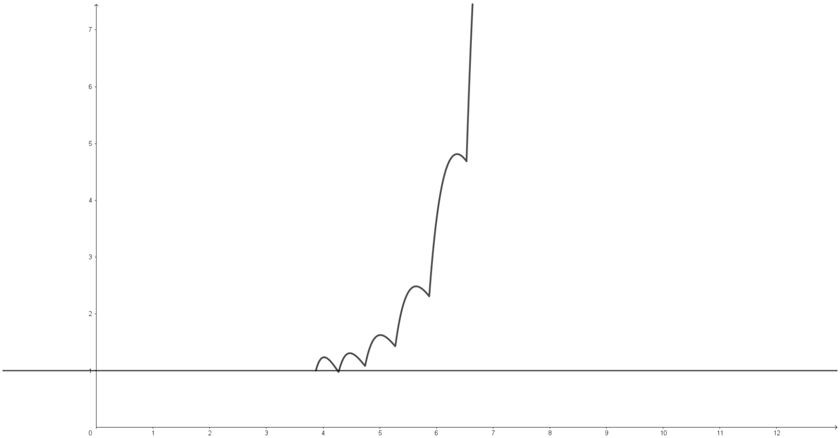

Figure: n=15

Video Assistant Referee for $n = 14$

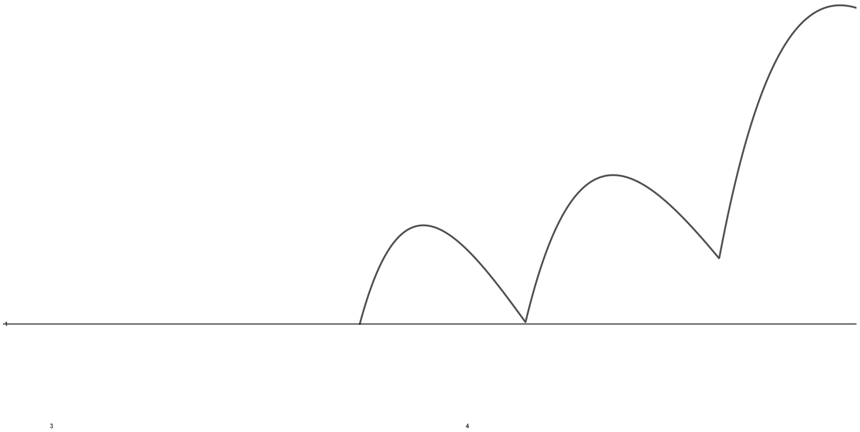

Figure: n=14

Video Assistant Referee for $n = 15$

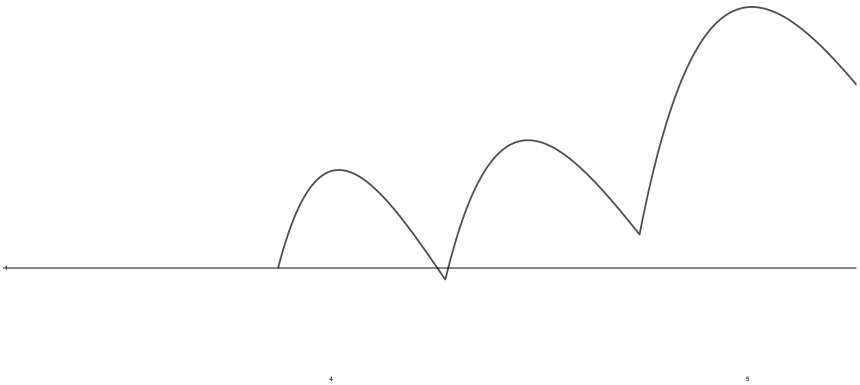

Figure: n=15

The n−th linear polarization constant of $\mathbb{R}^n$.

Now, let us revisit the idea proposed by A. Pappas and S. G. Révész: an appropriate choice of signs $\{\varepsilon_i\}_{i=1}^n$ that maximizes the Euclidean norm of $\sum_{i=1}^n \varepsilon_i v_i$.

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$$\left\| \sum_{i=1}^n \varepsilon_i v_i \right\|_2^2 = \sum_{i=1}^n \|\varepsilon_i v_i\|_2^2 + 2 \sum_{1\le i\neq j\le n} \varepsilon_i \varepsilon_j \langle v_i, v_j \rangle.$$

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If we consider the random vector of signs $(\varepsilon)_j = (\varepsilon_{j_1}, \ldots, \varepsilon_{j_n})$, where each sign is chosen independently with equal probability. The mean of the squared norm is

$$\frac{1}{2^n} \sum_{j=1}^{2^n} \left\| \sum_{i=1}^n \varepsilon_{j_i} v_i \right\|_2^2 = n.$$

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- Without loss of generality, we may assume that the choice of signs maximizing $\left\|\sum_{i=1}^{n} \varepsilon_i v_i\right\|_2^2$ is $(\varepsilon_1, \ldots, \varepsilon_n) = (1, \ldots, 1)$.

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- For this vector v, we have $\langle v, v\rangle \geq \langle v - 2v_i, v - 2v_i\rangle$, for all $1 \leq i \leq n$.
- Then,

$$\langle v_i, v\rangle \geq 1, \quad \text{for all } 1 \leq i \leq n.$$

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Theorem

Given $2 \leq n \leq 14$, *then* $c_n(\mathbb{R}^n) = \sqrt{n^n}$.

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- Take any set of unit vectors $\{v_i\}_{i=1}^n \subset \mathbb{R}^n$.
- Let $v = \sum_{i=1}^n v_i$ be the *longest sum* of them.

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- So, $\displaystyle\prod_{i=1}^n \left\langle v_i, \frac{v}{\|v\|} \right\rangle = f(a_1, \ldots, a_n) \geq \mu(\|v\|) \geq \frac{1}{\sqrt{n^n}}$.

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Theorem

For $2 \leq n \leq 14$, if $\{v_i\}_{i=1}^n \subset \mathbb{R}$ are unit vectors such that

$$\sup_{\|x\|=1} |\langle x, v_1 \rangle \cdots \langle x, v_n \rangle| = n^{-n/2},$$

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Proof

Recall that $\mu(s) = n^{-n/2} \Leftrightarrow s = \sqrt{n}$. Under the hypothesis, the *longest sum* $v = v_1 + \ldots + v_n$ has norm $\sqrt{n}$. But then, any other vector $v_{(\varepsilon)_j} = \sum_{i=1}^n \varepsilon_{j_i} v_i$ has norm $\sqrt{n}$, and it is easy to see that $\{v_i\}_{i=1}^n$ must be an orthonormal system.

Final Remarks

• For $n = 34$, M. Matolcsi and G. A. Muñoz gave an example in $\mathbb{R}^{34}$ where the *longest sum* v of some set of unit vectors $\{v_i\}_{i=1}^{34}$ does not satisfy the inequality $\prod |\langle v_i, v\rangle| \geq 34^{-17}$. From their example it is possible to construct many others for any $n > 34$.

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• Given $s \in [\sqrt{n}, n]$, if we denote by $\mathscr{F}(s)$ the set of all $n-$tuples of unit vectors $\{v_i\}_{i=1}^{n}$, such that its *longest sum* v has $\|v\| = s$, then the map

$$\Lambda : \mathscr{F}(s) \longrightarrow \Sigma_s$$

defined by

$$\Lambda(v_1, v_2, \ldots, v_n) = \frac{1}{s}\left(\langle v_1, v\rangle, \ldots, \langle v_n, v\rangle\right)$$

is not necessarily surjective. Then it is possible that the *longest sum* v still works as a good tester for some other values of $14 < n < 34$.

Thank You!