Existence and Properties of Frames of Iterations

Di Tella Workshop on Analysis and Beyond

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- $\{\phi_j\}_{j \in J} \subseteq \mathcal{H}$;
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We want to recover f from $\{\langle L^n f, \phi_j \rangle\}_{j \in J, n=0,1,2,\dots}$.

Stability: there exist $A, B > 0$ such that for all $f \in \mathcal{H}$

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A sequence $\{x_j\}_j \subseteq \mathcal{H}$ is a frame of $\mathcal{H}$ if there exist $A, B > 0$ such that for all $f \in \mathcal{H}$,

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$$\Updownarrow$$

$\{(L^*)^n \phi_j\}_{j,n}$ is a frame of $\mathcal{H}$ of iterations.

Given $\{v_j\}_{j\in J} \subseteq \mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$ *we study* frames of iterations:

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- If $T \in \mathcal{B}(\mathcal{H})$ which conditions on T guarantee that T admits a frame of itaration?
- If T admits a frame of iterations, what is the minimum number of generators we need to construct a frame of iterations of T?
- When do T and T^* (both) admit frames of iterations? If such frames exist, are they related in some way?

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- → A. Aldroubi, C. Cabrelli, U. Molter, and S. Tang, *Dynamical Sampling*, Appl. Comput. Harmon. Anal., 2017.
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- → C. Cabrelli, U. Molter, V. P., and F. Philipp, *Dynamical sampling on finite index sets*, J. Anal. Math., 2020.
  Finitely many generators and normal operators.

→ O. Christensen, M. Hasannasab, and F. Philipp, *Frame properties of operator orbits*, Math. Nachrichten, 2019.
Characterization of $\{T^n f : n \geq 0\}$ in terms of model subspace of the Hardy space in the unit disk.

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- C. Cabrelli, U. Molter, and D. Suarez, *Frames of iterations and vector valued model spaces*, in: Sampling, Approximation, and Signal Analysis (Harmonic Analysis in the Spirit of J. Rowland Higgins), 2023.
  Characterization of $\{T^n f : f \in \mathcal{F}, n \geq 0\}$ with $\mathcal{F}$ being at most countable in terms of model subspaces in Hardy spaces with multiplicity.

Hardy spaces

Hardy spaces

- Vector valued L^2-functions: For a Hilbert space $\mathcal{K}$,

$$L^2(\mathbb{T},\mathcal{K}) = \{f : \mathbb{T} \to \mathcal{K} : \int_{\mathbb{T}} \|f(z)\|^2_{\mathcal{K}}\, dz \ < \infty\}.$$

Equivalently,

$$f \in L^2(\mathbb{T},\mathcal{K}) \Longleftrightarrow f(z) = \sum_{k\in\mathbb{Z}} a_k z^k, \quad z \in \mathbb{T}, \ \sum_{k\in\mathbb{Z}} \|a_k\|^2_{\mathcal{K}} < \infty.$$

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- Hardy spaces with multiplicity: $f \in H^2_{\mathcal{K}} \iff f \in L^2(\mathbb{T},\mathcal{K})$ and

$$f(z) = \sum_{n\geq 0}^{\infty} a_n z^n, \quad z \in \mathbb{T}, \ \sum_{n\geq 0} \|a_n\|_{\mathcal{K}}^2 < \infty.$$

- Unilateral shift operator: It is the isometry $S : H^2_{\mathcal{K}} \to H^2_{\mathcal{K}}$

$$Sf(z) = zf(z), \quad z \in \mathbb{T}.$$

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- Given an orthonormal basis $\{\varepsilon_j\}_{j\in J}$ of $\mathcal{K}$,

$\{S^n \varepsilon_j : \ n \geq 0,\, j \in J\}$ is an orthonormal basis of $H^2_{\mathcal{K}}$.

- Basic frames of iterations : For any closed subspace $N \subseteq H^2_{\mathcal{K}}$

$$\{P_N(S^n \varepsilon_j) : \; n \geq 0, \, j \in J\} \qquad (\star)$$

is a Parseval frame (A=B=1) of N.

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- Similar frames of iterations:

Given $\mathcal{H}_1, \mathcal{H}_2$ two Hilbert spaces and $\{T_1^n v_j\}_{n\geqslant 0, j\in J} \subseteq \mathcal{H}_1$ and $\{T_2^n w_j\}_{n\geqslant 0, j\in J} \subseteq \mathcal{H}_2$ two frames of iterations, we say that they are similar if there exists an isomorphism $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$VT_1V^{-1} = T_2 \quad \text{and} \quad V(v_j) = w_j \; \forall \; j \in J.$$

We further say they are unitarily equivalent frames of iterations if V is unitary.

Theorem **(Christensen - Philipp - Hasannasab - Cabrelli - Molter - Suarez)**

Let $T \in \mathcal{B}(\mathcal{H})$ and $\{v_j\}_{j\in J} \subseteq \mathcal{H}$ such that $\{T^n v_j\}_{j\in J,\, n\geq 0}$ is a frame (Parseval frame).

Then it is similar (unitarily equivalent) to a basic frame

$$\{A_N^n(P_N\varepsilon_j)\}_{j\in J,\, n\geq 0},$$

where $N \subseteq H^2_{\ell^2(J)}$ is a model space and $\{\varepsilon_j\}_{j\in J}$ is the canonical basis of $\ell^2(J)$.

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- Note that:

→ $T \sim A_N$ (similar as operators)

→ $\|A_N\|_{op} \leq 1$

→ $A_N^* = S^*|_N$ and $(S^*)^n f \to 0$ as $n \to \infty$, for all $f \in N$.

- If $\|T\|_{op} \leq 1$, then T is a contraction.
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- Key result:

Theorem (Nikolski (1986))

Let $T \in \mathcal{B}(\mathcal{H})$. If T is a contraction and T^* is strongly stable, then there exists a Hilbert space $\mathcal{K}$ such that T is unitarily equivalent to A_N for some model space $N \subseteq H^2_{\mathcal{K}}$.

Theorem **(Aguilera - Cabrelli - Negreira - P. (2025)/ Cabrelli - Molter- P. - Philipp (2020))**

Let $T \in \mathcal{B}(\mathcal{H})$. Then,

- T admits a Parseval frame of iterations if and only if

$$\begin{cases} T & \text{is a contraction;} \\ T^* & \text{is strongly stable.} \end{cases}$$

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Index frame of an operator

- Let $T \in \mathcal{B}(\mathcal{H})$, we say that a collection of non-zero vectors $\{v_j\}_{j\in J} \subseteq \mathcal{H}$ is a set of generators for T if $\{T^n v_j\}_{j\in J, n\geqslant 0}$ is a frame of $\mathcal{H}$.

The frame index of T is

$$\boxed{\gamma(T) = \min\{d \in \mathbb{N} : \text{there exists a set of generators of d elements}\},}$$

if T admits a frame of iterations with a finite generating set. We say that $\gamma(T) = 0$ if T does not have a frame of iterations, and $\gamma(T) = +\infty$ if T admits a frame of iterations, but none with a finite generating set.

The Parseval frame index, $\gamma_p(T)$ has the same definition, but requiring that the frame of iterations is a Parseval frame.

Theorem (Aguilera - Cabrelli - Negreira - P. (2025))

Let $T \in \mathcal{B}(\mathcal{H})$ such that it admits a Parseval frame of iteration. Then

$$\gamma_p(T) = \dim \overline{(I_{\mathcal{H}} - TT^*)(\mathcal{H})}.$$

Moreover, the index $\gamma_p(T)$ is attained with linearly independent generators.

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- If T admits a frame of iterations, then

$$\rightarrow \gamma(T) = \min \{\gamma_p(Q) : Q \text{ contraction with PF of iterations}, Q \sim T\}.$$

Optimal Parseval frames of iterations for T and T^*

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Optimal Parseval frames of iterations for T and T^*

Let T be a contraction such that T and T^* are strongly stable $\Longrightarrow$ T and T^* admit Parseval frames of iterations.

Then:

- $\gamma_p(T) = \gamma_p(T^*)$;
- We can explictly construct “optimal” Parseval frames of iterations for T and T^*: $\{v_j\}_{j\in J}, \{w_j\}_{j\in J} \subseteq \mathcal{H}$ such that $\{T^n v_j\}_{j\in J}$ and $\{(T^*)^n w_j\}_{j\in J}$ are Parseval frames with $\gamma_p(T) = \#J = \gamma_p(T^*)$.

- T and T^* are strongly stable $\Longleftrightarrow$ there exists a model space $N \subseteq H^2_{\mathcal{K}}$ such that $T \sim A_N$ and $N^\perp$ is a "full range" S-invariant space (full range spaces were introduced by Helson in 1964).

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- We can characterize when Parseval frames of iteration of T and T^* are similar (similar as frames) in terms of the model spaces associated to T and T^*.

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A. Aguilera, C. Cabrelli, F. Negreira and V. P., *Existence and properties of frames of iterations*, (2025), `https://arxiv.org/abs/2506.00567`.

¡Muchas gracias!