

Multiplicative Toeplitz operators and duality on Hardy spaces of Dirichlet series

Santiago Muro

Universidad Nacional de Rosario, Argentina
and CONICET

Joint work with **Matías Palumbo** (Universidad Nacional de Rosario) and
Rodrigo Cardeccia (Instituto Balseiro, Bariloche, Argentina)

June 18, 2025
Analysis and Beyond

(Our) motivation

An important result in linear dynamics:

Godefroy-Shapiro (1991)

Let M_φ^* be the adjoint of a non-constant multiplication operator on the Hardy space of analytic functions $H_2(\mathbb{D})$.

Then M_φ^* is *hypercyclic* (and *chaotic*) if and only if $\varphi(\mathbb{D}) \cap \mathbb{T}$.

(Our) motivation

An important result in linear dynamics:

Godefroy-Shapiro (1991)

Let M_φ^* be the adjoint of a non-constant multiplication operator on the Hardy space of analytic functions $H_2(\mathbb{D})$.

Then M_φ^* is *hypercyclic* (and *chaotic*) if and only if $\varphi(\mathbb{D}) \cap \mathbb{T}$.

Is it true for Hardy spaces of Dirichlet series?

$$\sum_{n \geq 1} a_n n^{-s} = a_1 + \sum_{n \geq 2} a_n n^{-s}, \quad s \in \mathbb{C}.$$

(Our) motivation

An important result in linear dynamics:

Godefroy-Shapiro (1991)

Let M_φ^* be the adjoint of a non-constant multiplication operator on the Hardy space of analytic functions $H_2(\mathbb{D})$.

Then M_φ^* is *hypercyclic* (and *chaotic*) if and only if $\varphi(\mathbb{D}) \cap \mathbb{T}$.

Is it true for Hardy spaces of Dirichlet series?

$$\sum_{n \geq 1} a_n n^{-s} = a_1 + \sum_{n \geq 2} a_n n^{-s}, \quad s \in \mathbb{C}.$$

Godefroy-Shapiro Theorem can be extended to some reflexive spaces of analytic functions on a region R with the hypothesis

$$\star \quad \|M_\varphi\| = \sup_R |\varphi|$$

Hardy Space H_2 on the Disc

Let $H_2(\mathbb{D})$ be the Hilbert space of analytic functions on the unit disc with square-summable Taylor coefficients:

$$H_2(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

$$\|f\|_{H_2}^2 = \sum_{n=0}^{\infty} |a_n|^2 = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/2}$$

H_2 on the Torus: Harmonic Counterpart

Link between analytic functions on the disc and one-sided Fourier series:

$H_2(\mathbb{T})$ is closure in $L^2(\mathbb{T})$ of the analytic trigonometric polynomials:

$$H_2(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0 \right\}$$

Fourier coefficients: $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt$

H_2 on the Torus: Harmonic Counterpart

Link between analytic functions on the disc and one-sided Fourier series:

$H_2(\mathbb{T})$ is closure in $L^2(\mathbb{T})$ of the analytic trigonometric polynomials:

$$H_2(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0 \right\}$$

Fourier coefficients: $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt$

$H_2(\mathbb{T})$ is isometrically isomorphic to $H_2(\mathbb{D})$:

- $H_2(\mathbb{D}) \rightarrow H_2(\mathbb{T})$ via radial limits of analytic functions.
- Poisson operator: $H_2(\mathbb{T}) \rightarrow H_2(\mathbb{D})$, $\sum a_n e^{int} \mapsto \sum a_n z^n$

Toeplitz Operators on H_2

$H_\infty(\mathbb{D})$ = Banach space of bounded analytic functions

$$H_\infty(\mathbb{D}) \stackrel{1}{=} H_\infty(\mathbb{T}) \hookrightarrow L_\infty(\mathbb{T})$$

- $M_\phi f = \phi f$ multiplication operator on $H_2(\mathbb{D}) \Leftrightarrow \phi \in H_\infty(\mathbb{D})$,

Toeplitz Operators on H_2

$H_\infty(\mathbb{D})$ = Banach space of bounded analytic functions

$$H_\infty(\mathbb{D}) \stackrel{1}{=} H_\infty(\mathbb{T}) \hookrightarrow L_\infty(\mathbb{T})$$

- $M_\phi f = \phi f$ multiplication operator on $H_2(\mathbb{D}) \Leftrightarrow \phi \in H_\infty(\mathbb{D})$,
- for any $\phi \in L_\infty(\mathbb{T})$, **Toeplitz operator**: T_ϕ

$$\begin{array}{ccccc} H_2(\mathbb{T}) & \rightarrow & L_2(\mathbb{T}) & \rightarrow & H_2(\mathbb{T}) \\ f & \mapsto & \phi f & \mapsto & T_\phi f := P_+(\phi f) \end{array}$$

- **Riesz projection**: $P_+ : L_2(\mathbb{T}) \rightarrow H_2(\mathbb{T}) \leftarrow$ orthogonal projection

Toeplitz Operators on H_2

$H_\infty(\mathbb{D}) =$ Banach space of bounded analytic functions

$$H_\infty(\mathbb{D}) \stackrel{1}{=} H_\infty(\mathbb{T}) \hookrightarrow L_\infty(\mathbb{T})$$

- $M_\phi f = \phi f$ multiplication operator on $H_2(\mathbb{D}) \Leftrightarrow \phi \in H_\infty(\mathbb{D})$,
- for any $\phi \in L_\infty(\mathbb{T})$, **Toeplitz operator**: T_ϕ

$$\begin{array}{ccccc} H_2(\mathbb{T}) & \rightarrow & L_2(\mathbb{T}) & \rightarrow & H_2(\mathbb{T}) \\ f & \mapsto & \phi f & \mapsto & T_\phi f := P_+(\phi f) \end{array}$$

- **Riesz projection**: $P_+ : L_2(\mathbb{T}) \rightarrow H_2(\mathbb{T}) \leftarrow$ orthogonal projection
- T_ϕ bounded iff $\phi \in L_\infty(\mathbb{T})$. Moreover, $\|T_\phi\| = \|\phi\|_\infty$
- Matrix representation: $T_\phi = (a_{i,j})_{i,j \geq 0} = (a_{i-j})_{i,j \geq 0}$

Toeplitz Operators on H_2

$H_\infty(\mathbb{D}) =$ Banach space of bounded analytic functions

$$H_\infty(\mathbb{D}) \stackrel{1}{=} H_\infty(\mathbb{T}) \hookrightarrow L_\infty(\mathbb{T})$$

- $M_\phi f = \phi f$ multiplication operator on $H_2(\mathbb{D}) \Leftrightarrow \phi \in H_\infty(\mathbb{D})$,
- for any $\phi \in L_\infty(\mathbb{T})$, **Toeplitz operator**: T_ϕ

$$\begin{array}{ccccc} H_2(\mathbb{T}) & \rightarrow & L_2(\mathbb{T}) & \rightarrow & H_2(\mathbb{T}) \\ f & \mapsto & \phi f & \mapsto & T_\phi f := P_+(\phi f) \end{array}$$

- **Riesz projection**: $P_+ : L_2(\mathbb{T}) \rightarrow H_2(\mathbb{T}) \leftarrow$ orthogonal projection
- T_ϕ bounded iff $\phi \in L_\infty(\mathbb{T})$. Moreover, $\|T_\phi\| = \|\phi\|_\infty$
- Matrix representation: $T_\phi = (a_{i,j})_{i,j \geq 0} = (a_{i-j})_{i,j \geq 0}$

Co-analytic Toeplitz operators on H_2 : $T_{\bar{\phi}}$ for $\phi \in H_\infty(\mathbb{D})$

$$T_{\bar{\phi}} = M_\phi^* \quad \leftarrow \text{co-analytic Toeplitz} \equiv \text{adjoint of multiplier}$$

$$\langle T_{\bar{\phi}} f, g \rangle_{H_2} = \langle f, M_\phi g \rangle_{H_2}$$

Hardy Spaces for $p \neq 2$

- $H_p(\mathbb{T})$: Closure of analytic trigonometric polynomials in $L_p(\mathbb{T})$

$$H_p(\mathbb{T}) \stackrel{1}{=} H_p(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{H_p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty \right\}$$

- Multipliers = $H_\infty(\mathbb{D})$
- **Riesz projection** $P_+ : L_p(\mathbb{T}) \rightarrow H_p(\mathbb{T})$:
 - **Bounded** for $1 < p < \infty$ (M. Riesz theorem)
Not a contraction for $p \neq 2$, Hollenbeck, Verbitsky '00: $\|P_+\| = \frac{1}{\sin(\pi/p)}$
- **Toeplitz operators**: T_ϕ bounded iff $\phi \in L_\infty(\mathbb{T})$.
- **Duality**: $(H_p)^* \cong H_q$ ($\frac{1}{p} + \frac{1}{q} = 1$) \Leftarrow but **not isometric** for $p \neq 2$
- Adjoint multiplier vs co-analytic Toeplitz:

$$M_\phi^* : (H_p)^* \rightarrow (H_p)^* \qquad T_{\bar{\phi}} : H_q \rightarrow H_q$$

$$\text{for } p \neq 2, \quad \star \|M_\phi^*\| = \|\phi\|_\infty, \quad \text{but } \neq \|T_{\bar{\phi}}\|_{H_q \rightarrow H_q}$$

Hardy Spaces of Dirichlet Series

- Dirichlet series: $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, where $s = \sigma + it \in \mathbb{C}$
- $1 < p < \infty$, **Hardy space** \mathcal{H}_p : Completion of Dirichlet polynomials under Besicovitch norm

$$\|D\|_{\mathcal{H}_p} = \lim_{R \rightarrow \infty} \left(\frac{1}{2R} \int_{-R}^R |D(it)|^p dt \right)^{1/p}$$

- \mathcal{H}_∞ : Dirichlet series representing bounded analytic functions in $\text{Re } s > 0$

Multipliers of \mathcal{H}_p (Hedenmalm-Lindqvist-Seip/Bayart)

$$\mathcal{M}(\mathcal{H}_p) = \mathcal{H}_\infty, \quad \text{with } \|M_D\| = \|D\|_\infty$$

Hardy Spaces of Dirichlet Series

- Dirichlet series: $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, where $s = \sigma + it \in \mathbb{C}$
- $1 < p < \infty$, **Hardy space** \mathcal{H}_p : Completion of Dirichlet polynomials under Besicovitch norm

$$\|D\|_{\mathcal{H}_p} = \lim_{R \rightarrow \infty} \left(\frac{1}{2R} \int_{-R}^R |D(it)|^p dt \right)^{1/p}$$

- \mathcal{H}_{∞} : Dirichlet series representing bounded analytic functions in $\text{Re } s > 0$

Multipliers of \mathcal{H}_p (Hedenmalm-Lindqvist-Seip/Bayart)

$$\mathcal{M}(\mathcal{H}_p) = \mathcal{H}_{\infty}, \quad \text{with } \|M_D\| = \|D\|_{\infty}$$

- $p = 2$: \mathcal{H}_2 the Hilbert space of Dirichlet series with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$
- $1 < p < \infty$, Aleman '14: \mathcal{H}_p is reflexive with Schauder basis $(n^{-s})_n$

Hardy Spaces of Dirichlet Series

- Dirichlet series: $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, where $s = \sigma + it \in \mathbb{C}$
- $1 < p < \infty$, **Hardy space** \mathcal{H}_p : Completion of Dirichlet polynomials under Besicovitch norm

$$\|D\|_{\mathcal{H}_p} = \lim_{R \rightarrow \infty} \left(\frac{1}{2R} \int_{-R}^R |D(it)|^p dt \right)^{1/p}$$

- \mathcal{H}_{∞} : Dirichlet series representing bounded analytic functions in $\text{Re } s > 0$

Multipliers of \mathcal{H}_p (Hedenmalm-Lindqvist-Seip/Bayart)

$$\mathcal{M}(\mathcal{H}_p) = \mathcal{H}_{\infty}, \quad \text{with } \|M_D\| = \|D\|_{\infty}$$

- $p = 2$: \mathcal{H}_2 the Hilbert space of Dirichlet series with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$
- $1 < p < \infty$, Aleman '14: \mathcal{H}_p is reflexive with Schauder basis $(n^{-s})_n$

Remark: Dirichlet series in \mathcal{H}_{∞} converge in $\text{Re } s > 0$

but \mathcal{H}_p converge in $\text{Re } s > 1/2 \Leftarrow \mathcal{H}_p$ does not satisfy hypothesis \star

(Multiplicative) Toeplitz Operators on \mathcal{H}_2

- Besicovitch space: $B_{\mathbb{Q}_+}^2$
Hilbert space of series $\sum_{q \in \mathbb{Q}_+} b_q q^{-s}$, with $\sum_q |b_q|^2 < \infty$.
- Multiplicative Toeplitz operator for $D = \sum_{q \in \mathbb{Q}_+} b_q q^{-s}$:

$$T_D : \mathcal{H}_2 \rightarrow \mathcal{H}_2, \quad T_D E = P_{\mathbb{N}}(D \cdot E)$$

where $P_{\mathbb{N}}$ is the Riesz projection (orthogonal projection) to the \mathbb{N} -coefficients

- Matrix representation: $T_D = (a_{n,m})_{n,m \geq 1} = (a_{n/m})_{n,m \geq 1}$

Like in the classical case:

Co-analytic Toeplitz \equiv adjoint of co-analytic multipliers

For $D(s) = \sum_{n \in \mathbb{N}} a_n \left(\frac{1}{n}\right)^{-s} = \sum_{n \in \mathbb{N}} a_n n^s$ (“co-analytic Dirichlet series”):

$$T_D = M_{jD}^* : \mathcal{H}_2 \rightarrow \mathcal{H}_2$$

where $jD = \sum \overline{a_n} n^{-s}$

(Multiplicative) Toeplitz Operators on \mathcal{H}_p , $p \neq 2$

- Is it possible to define Multiplicative Toeplitz operators? For symbols of the form $D = \sum_{q \in \mathbb{Q}_+} b_q q^{-s}$:

$$T_D : \mathcal{H}_p \rightarrow \mathcal{H}_p, \quad T_D E = P_{\mathbb{N}}(D \cdot E)$$

(Multiplicative) Toeplitz Operators on \mathcal{H}_p , $p \neq 2$

- Is it possible to define Multiplicative Toeplitz operators? For symbols of the form $D = \sum_{q \in \mathbb{Q}_+} b_q q^{-s}$:

$$T_D : \mathcal{H}_p \rightarrow \mathcal{H}_p, \quad T_D E = P_{\mathbb{N}}(D \cdot E)$$

- For which symbols?
- $P_{\mathbb{N}}$ projection to the \mathbb{N} -coefficients is not bounded!
- Co-analytic Toeplitz vs adjoint of analytic multipliers?
- duality issue: $(\mathcal{H}_p)^* \neq \mathcal{H}_q$? Problem: describe $(\mathcal{H}_p)^*$

(Multiplicative) Toeplitz Operators on \mathcal{H}_p , $p \neq 2$

- Is it possible to define Multiplicative Toeplitz operators? For symbols of the form $D = \sum_{q \in \mathbb{Q}_+} b_q q^{-s}$:

$$T_D : \mathcal{H}_p \rightarrow \mathcal{H}_p, \quad T_D E = P_{\mathbb{N}}(D \cdot E)$$

- For which symbols?
- $P_{\mathbb{N}}$ projection to the \mathbb{N} -coefficients is not bounded!
- Co-analytic Toeplitz vs adjoint of analytic multipliers?
- duality issue: $(\mathcal{H}_p)^* \neq \mathcal{H}_q$? Problem: describe $(\mathcal{H}_p)^*$
- what is the harmonic counterpart of \mathcal{H}_p ?
- (Saksman-Seip '09) There are series $D \in \mathcal{H}_\infty \subset \mathcal{H}_p$ such that

$$\lim_{R \rightarrow \infty} \left(\frac{1}{2R} \int_{-R}^R |D(it)|^p dt \right) \text{ does not exist}$$

Step toward a description of the dual Space \mathcal{H}_p^*

$1 < p < \infty$. We denote $(n^s)_{n \in \mathbb{N}}$ the dual basis of $(n^{-s})_{n \in \mathbb{N}}$, so that We identify \mathcal{H}_p^* with the Banach space of formal co-analytic series $D = \sum_{n=1}^{\infty} d_n n^s$ such that

$$\sum_{n=1}^{\infty} d_n e_n \text{ converges for every } E = \sum_{n=1}^{\infty} e_n n^{-s} \in \mathcal{H}_p$$

The norm is the dual norm

$$\|D\|_{\mathcal{H}_p^*} = \sup_{\|E\|_{\mathcal{H}_p} \leq 1} |\langle D, E \rangle| := \sup_{\|E\|_{\mathcal{H}_p} \leq 1} \left| \sum_{n=1}^{\infty} d_n e_n \right|.$$

- where do series in \mathcal{H}_p^* converge? They converge on left-half planes.
In the \mathcal{H}_p case: $\sigma_c(\mathcal{H}_p) = \frac{1}{2}$

\mathcal{H}_p^* as co-analytic series

- where do series in \mathcal{H}_p^* converge?

Theorem: $\sigma_c(\mathcal{H}_p^*) = -\frac{1}{2}$.

The proof is related to the question: which translations of ζ are in \mathcal{H}_p ?

\mathcal{H}_p^* as co-analytic series

- where do series in \mathcal{H}_p^* converge?

Theorem: $\sigma_c(\mathcal{H}_p^*) = -\frac{1}{2}$.

The proof is related to the question: which translations of ζ are in \mathcal{H}_p ?

- Absolute convergence? $\sigma_a(\mathcal{H}_p) = \frac{1}{2} \leftarrow$ proof needs other tools

\mathcal{H}_p^* as co-analytic series

- where do series in \mathcal{H}_p^* converge?

Theorem: $\sigma_c(\mathcal{H}_p^*) = -\frac{1}{2}$.

The proof is related to the question: which translations of ζ are in \mathcal{H}_p ?

- Absolute convergence? $\sigma_a(\mathcal{H}_p) = \frac{1}{2} \Leftarrow$ proof needs other tools
 - For $p < 2$ easy: $\mathcal{H}_p^* \hookrightarrow \mathcal{H}_2^* \Rightarrow \sigma_a(\mathcal{H}_p^*) \geq \sigma_a(\mathcal{H}_2^*) = -\frac{1}{2}$, but $-\frac{1}{2} = \sigma_c(\mathcal{H}_p^*) \geq \sigma_a(\mathcal{H}_p^*)$. Thus, $\sigma_a(\mathcal{H}_p^*) = -\frac{1}{2}$
 - For $p > 2$: Monomial convergence

\mathcal{H}_p , Harmonic Counterpart

- **Harmonic Counterpart:** functions of infinitely many variables
 $n \in \mathbb{N} \Rightarrow n = p^\alpha := p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_k^{\alpha_k}$, where $p = (p_1, p_2, \dots)$
sequence of prime numbers

\mathcal{H}_p , Harmonic Counterpart

- **Harmonic Counterpart:** functions of infinitely many variables
 $n \in \mathbb{N} \Rightarrow n = p^\alpha := p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_k^{\alpha_k}$, where $p = (p_1, p_2, \dots)$
sequence of prime numbers

Bohr Transform (1913): $n^{-s} = (p^\alpha)^{-s} \leftrightarrow z^\alpha = z_1^{\alpha_1} \dots z_k^{\alpha_k}$

\mathcal{H}_p , Harmonic Counterpart

- **Harmonic Counterpart:** functions of infinitely many variables
 $n \in \mathbb{N} \Rightarrow n = p^\alpha := p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_k^{\alpha_k}$, where $p = (p_1, p_2, \dots)$
sequence of prime numbers

$$\text{Bohr Transform (1913): } n^{-s} = (p^\alpha)^{-s} \leftrightarrow z^\alpha = z_1^{\alpha_1} \dots z_k^{\alpha_k}$$



Dirichlet series \leftrightarrow analytic functions \leftrightarrow Fourier series
infinite variables \qquad infinite variables

- Formally, Bohr's correspondence:

$$\mathfrak{B} : \sum_{\alpha} c_{\alpha} z^{\alpha} \mapsto \sum_{n=1}^{\infty} a_n n^{-s}$$

where $c_{\alpha} = a_n$ for $n = p^{\alpha}$.

Finite-dimensional Case

- **N -Dimensional polydisc and torus ($H_p(\mathbb{D}^N)$ and $H_p(\mathbb{T}^N)$):**
 - $H_p(\mathbb{D}^N)$ functions analytic on \mathbb{D}^N , satisfying integral mean conditions on p -norms.
 - $H_p(\mathbb{T}^N)$ functions on $L_p(\mathbb{T}^N)$ with vanishing Fourier coefficients for multi-indices outside \mathbb{Z}_+^N .
 - Poisson operator: $H_p(\mathbb{T}^N) \stackrel{1}{=} H_p(\mathbb{D}^N)$
- Multipliers: $M_\phi, \phi \in H_\infty(\mathbb{D}^N)$
- Riesz projection: $P_+^{(N)} : L_p(\mathbb{T}^N) \rightarrow H_p(\mathbb{T}^N)$,
 $\|P_+^{(N)}\| \rightarrow \infty$ as $N \rightarrow \infty$ for $p \neq 2$
- Toeplitz operators T_ϕ , bounded iff $\phi \in L_\infty(\mathbb{T}^N)$
- Duality: $H_p(\mathbb{T}^N)^* = H_q(\mathbb{T}^N)$ \leftarrow equivalent norms ($\rightarrow \infty$ as $N \rightarrow \infty$)
- Co-analytic Toeplitz operator = Adjoint of multipliers \leftarrow equivalent norms

Infinite-Dimensional Case: Fourier series & analytic functions

- $H_p(\mathbb{T}^\infty) = \left\{ f \in L_p(\mathbb{T}^\infty) : \int_{\mathbb{T}^\infty} f(w) e^{-i\alpha w} dw = 0 \forall \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \right\}$
- $H_p(\mathbb{D}^\infty)$: **formal power series** $f = \sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} c_\alpha z^\alpha$
 - 1 for each $N \in \mathbb{N}$, $f_N(z) = f(z_1, \dots, z_N, 0, \dots) \in H_p(\mathbb{D}^N)$ and
 - 2 $\|f\|_{H_p(\mathbb{D}^\infty)} := \sup_{N \in \mathbb{N}} \|f_N\|_{H_p(\mathbb{D}^N)} = \sup_{N \in \mathbb{N}} \sup_{0 < r < 1} \left(\int_{\mathbb{T}^N} |f(rw)|^p dw \right)^{\frac{1}{p}} < \infty.$
- 1 + 2 \Leftarrow Hilbert criterion

Infinite-Dimensional Case: Fourier series & analytic functions

- $H_p(\mathbb{T}^\infty) = \left\{ f \in L_p(\mathbb{T}^\infty) : \int_{\mathbb{T}^\infty} f(w) e^{-i\alpha w} dw = 0 \forall \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \right\}$
- $H_p(\mathbb{D}^\infty)$: **formal power series** $f = \sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} c_\alpha z^\alpha$
 - 1 for each $N \in \mathbb{N}$, $f_N(z) = f(z_1, \dots, z_N, 0, \dots) \in H_p(\mathbb{D}^N)$ and
 - 2 $\|f\|_{H_p(\mathbb{D}^\infty)} := \sup_{N \in \mathbb{N}} \|f_N\|_{H_p(\mathbb{D}^N)} = \sup_{N \in \mathbb{N}} \sup_{0 < r < 1} \left(\int_{\mathbb{T}^N} |f(rw)|^p dw \right)^{\frac{1}{p}} < \infty.$
- 1 + 2 \Leftarrow Hilbert criterion
- Power series $f = \sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} c_\alpha z^\alpha \in H_p(\mathbb{D}^\infty)$ converge to analytic function (*monomial convergence*) on $\mathbb{D}_2^\infty = B_{C_0} \cap \ell_2 \subset \ell_2$ (Bayart, Defant, ...)

Infinite-Dimensional Case: Fourier series & analytic functions

- $H_p(\mathbb{T}^\infty) = \left\{ f \in L_p(\mathbb{T}^\infty) : \int_{\mathbb{T}^\infty} f(w) e^{-i\alpha w} dw = 0 \forall \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \right\}$

- $H_p(\mathbb{D}^\infty)$: **formal power series** $f = \sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} c_\alpha z^\alpha$

① for each $N \in \mathbb{N}$, $f_N(z) = f(z_1, \dots, z_N, 0, \dots) \in H_p(\mathbb{D}^N)$ and

② $\|f\|_{H_p(\mathbb{D}^\infty)} := \sup_{N \in \mathbb{N}} \|f_N\|_{H_p(\mathbb{D}^N)} = \sup_{N \in \mathbb{N}} \sup_{0 < r < 1} \left(\int_{\mathbb{T}^N} |f(rw)|^p dw \right)^{\frac{1}{p}} < \infty.$

① + ② \Leftarrow Hilbert criterion

- Power series $f = \sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} c_\alpha z^\alpha \in H_p(\mathbb{D}^\infty)$ converge to analytic function (*monomial convergence*) on $\mathbb{D}_2^\infty = B_{C_0} \cap \ell_2 \subset \ell_2$ (Bayart, Defant, ...)
- All three spaces are isometrically isomorphic:

Fourier series $\xrightarrow[\text{(Poisson)}]{P_\infty}$ analytic functions $\xrightarrow[\text{(Bohr)}]{\mathfrak{B}}$ Dirichlet series
infinite variables infinite variables

$$H_p(\mathbb{T}^\infty) \quad \leftrightarrow \quad H_p(\mathbb{D}_2^\infty) \quad \leftrightarrow \quad \mathcal{H}_p$$

Case $p = \infty$: $H_\infty(\mathbb{T}^\infty) \leftrightarrow H_\infty(B_{C_0}) \leftrightarrow \mathcal{H}_\infty \Leftarrow$ spaces of multipliers

Infinite-Dimensional case: Toeplitz operators T_ϕ

- Riesz projection P_+

$$P_+ : \sum_{\alpha \in \mathbb{Z}^{(\mathbb{N})}} a_\alpha w^\alpha \in L_p(\mathbb{T}^\infty) \mapsto \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha w^\alpha \in H_p(\mathbb{T}^\infty)$$

bounded on $L_p(\mathbb{T}^\infty)$ **only for $p = 2$** orthogonal

- for $p = 2$: T_ϕ bounded Toeplitz operator on $H_2(\mathbb{T}^\infty)$ iff $\phi \in L_\infty(\mathbb{T}^\infty)$
 - $\|T_\phi\| = \|\phi\|_\infty$
 - Toeplitz operators on $H_2(\mathbb{T}^\infty)$ correspond to **multiplicative Toeplitz operators** on \mathcal{H}_2 .


Infinite-Dimensional case: Toeplitz operators T_ϕ

- Riesz projection P_+

$$P_+ : \sum_{\alpha \in \mathbb{Z}^{(\mathbb{N})}} a_\alpha w^\alpha \in L_p(\mathbb{T}^\infty) \mapsto \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha w^\alpha \in H_p(\mathbb{T}^\infty)$$

bounded on $L_p(\mathbb{T}^\infty)$ **only for $p = 2$** \leftarrow orthogonal

- for $p = 2$: T_ϕ bounded Toeplitz operator on $H_2(\mathbb{T}^\infty)$ iff $\phi \in L_\infty(\mathbb{T}^\infty)$
 - $\|T_\phi\| = \|\phi\|_\infty$
 - Toeplitz operators on $H_2(\mathbb{T}^\infty)$ correspond to **multiplicative Toeplitz operators** on \mathcal{H}_2 .
- for $p \neq 2$ we cannot define P_+

$$f \in H_p(\mathbb{T}^\infty) \longrightarrow \phi f = \sum_{\alpha \in \mathbb{Z}^{(\mathbb{N})}} a_\alpha w^\alpha \in L_p(\mathbb{T}^\infty) \xrightarrow{P_+} \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha w^\alpha \in H_p(\mathbb{T}^\infty)$$



Infinite-Dimensional case: Toeplitz operators T_ϕ

- Riesz projection P_+

$$P_+ : \sum_{\alpha \in \mathbb{Z}^{(\mathbb{N})}} a_\alpha w^\alpha \in L_p(\mathbb{T}^\infty) \mapsto \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha w^\alpha \in H_p(\mathbb{T}^\infty)$$

bounded on $L_p(\mathbb{T}^\infty)$ **only for $p = 2$** \leftarrow orthogonal

- for $p = 2$: T_ϕ bounded Toeplitz operator on $H_2(\mathbb{T}^\infty)$ iff $\phi \in L_\infty(\mathbb{T}^\infty)$
 - $\|T_\phi\| = \|\phi\|_\infty$
 - Toeplitz operators on $H_2(\mathbb{T}^\infty)$ correspond to **multiplicative Toeplitz operators** on \mathcal{H}_2 .
- for $p \neq 2$ we cannot define P_+

$$f \in H_p(\mathbb{T}^\infty) \longrightarrow \phi f = \sum_{\alpha \in \mathbb{Z}^{(\mathbb{N})}} a_\alpha w^\alpha \in L_p(\mathbb{T}^\infty) \xrightarrow{P_+} \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha w^\alpha \in H_p(\mathbb{T}^\infty)$$


- characterize symbols of Toeplitz operators on $H_p(\mathbb{D}_2^\infty)$
- understand \mathcal{H}_p^* or $H_p(\mathbb{D}_2^\infty)^*$: see them as co-analytic series/functions

Extended Bohr transform

$$\mathcal{B} : \text{Trigonometric Series} \xrightarrow{(a_q := c_{p^\alpha})} \text{Dirichlet Series}$$

$$\begin{array}{l} \sum_{\alpha \in \mathbb{Z}^{(\mathbb{N})}} c_\alpha w^\alpha \\ \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha w^{-\alpha} \end{array} \qquad \begin{array}{l} \sum_{q \in \mathbb{Q}_+} a_n n^{-s} \\ \sum_{n \in \mathbb{N}} a_n n^s. \end{array}$$

$P_\infty \mathcal{B}^{-1}$ is a surjective isometry between \mathcal{H}_p^* and $H_p(\mathbb{D}_2^\infty)^*$

$$\sum_{n=1}^{\infty} a_n n^s \in \mathcal{H}_p^* \xrightarrow{P_\infty \mathcal{B}^{-1}} \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_{p^\alpha} \bar{z}^\alpha \in H_p(\mathbb{D}_2^\infty)^*. \quad (1)$$

Theorem: Hilbert type criterion

$$\varphi = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha \bar{z}^\alpha \in H_p(\mathbb{D}_2^\infty)^* \Leftrightarrow \text{for every } N \in \mathbb{N}:$$

$$\varphi_N := \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha \bar{z}^\alpha \in H_p(\mathbb{D}^N)^* \text{ and } \sup_{N \in \mathbb{N}} \|\varphi_N\|_{H_p(\mathbb{D}^N)^*} < \infty.$$

In this case, $\|\varphi\| = \sup_{N \in \mathbb{N}} \|i_N^*(\varphi)\|_{H_p(\mathbb{D}^N)^*}$ and $\pi_N^*(\varphi_N) \rightarrow \varphi$ in $H_p(\mathbb{D}^N)^*$.

Monomial convergence

Theorem (Set of monomial convergence)

Co-analytic series in $H_p(\mathbb{D}_2^\infty)^*$ converge absolutely in \mathbb{D}_2^∞

Corollary

Let $1 < p < \infty$. Then, $\sigma_a(\mathcal{H}_p^*) = -\frac{1}{2}$.

Proof.

We already know $\sigma_a(\mathcal{H}_p^*) \leq \sigma_c(\mathcal{H}_p^*) = -\frac{1}{2}$.

For the reverse inequality, let $D = \sum_{n=1}^{\infty} a_n n^s \in \mathcal{H}_p^*$ and $\operatorname{Re}(s) < -\frac{1}{2}$. Then

- $p^s \in B_{c_0} \cap \ell_2 = \mathbb{D}_2^\infty = \operatorname{mon}(H_p(\mathbb{D}_2^\infty)^*)$
- $P_\infty \mathcal{B}^{-1}(D) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_{p^\alpha} \bar{z}^\alpha \in H_p(\mathbb{D}_2^\infty)^*$

So $\sum_{n=1}^{\infty} |a_n n^s| = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_{p^\alpha}| (p^{\operatorname{Re}(s)})^\alpha < \infty$ □

Multiplicative Toeplitz operators, $T_D : E \mapsto P_{\mathbb{N}}(D \cdot E)$

$D = \sum_{n \in \mathbb{N}} a_n n^s$ a co-analytic Dirichlet series $\Leftarrow T_D$ is a co-analytic multiplicative Toeplitz operator.

Co-analytic multiplicative Toeplitz operators \neq adjoint of multipliers of \mathcal{H}_p

Remark

Co-analytic multiplicative Toeplitz operators \leftrightarrow multipliers of \mathcal{H}_p^* .

$T_D = M_D^*$ for $M_D : \mathcal{H}_p^* \rightarrow \mathcal{H}_p^*$ the multiplication operator by D .

If $D = \sum_{k=1}^{\infty} d_k k^s$, and $n, m \in \mathbb{N}$ then

$$\langle M_D^* n^{-s}, m^s \rangle_{\mathcal{H}_p, \mathcal{H}_p^*} = \langle n^{-s}, D m^s \rangle = \langle n^{-s}, \sum_{k=1}^{\infty} d_k (k \cdot m)^s \rangle = \sum_{k=1}^{\infty} d_k \delta_{n, k \cdot m} = d_{\frac{n}{m}},$$

while since $T_D n^{-s} = P_{\mathbb{N}} \left(\sum_{k=1}^{\infty} d_k \left(\frac{n}{k}\right)^{-s} \right) = \sum_{\substack{k \in \mathbb{N} \\ k|n}} d_k \left(\frac{n}{k}\right)^{-s}$,

$$\langle T_D n^{-s}, m^s \rangle = \left\langle \sum_{\substack{k \in \mathbb{N} \\ k|n}} d_k \left(\frac{n}{k}\right)^{-s}, m^s \right\rangle = \sum_{k=1}^{\infty} d_k \delta_{\frac{n}{k}, m} = d_{\frac{n}{m}}.$$

$\mathcal{T}_p = \{D = \sum_{q \in \mathbb{Q}_+} a_q q^{-s} \text{ define bounded Toeplitz operators on } \mathcal{H}_p\}$

Can we describe \mathcal{T}_p ? What are the multipliers of \mathcal{H}_p^* ?

$\mathcal{T}_p = \{D = \sum_{q \in \mathbb{Q}_+} a_q q^{-s} \text{ define bounded Toeplitz operators on } \mathcal{H}_p\}$

Can we describe \mathcal{T}_p ? What are the multipliers of \mathcal{H}_p^* ?

For $p = 2$: $\mathcal{T}_2 = \mathcal{B}(L_\infty(\mathbb{T}^\infty))$. Multipliers of $\mathcal{H}_2^* = \int \mathcal{H}_\infty$

$\mathcal{T}_p = \{D = \sum_{q \in \mathbb{Q}_+} a_q q^{-s} \text{ define bounded Toeplitz operators on } \mathcal{H}_p\}$

Can we describe \mathcal{T}_p ? What are the multipliers of \mathcal{H}_p^* ?

For $p = 2$: $\mathcal{T}_2 = \mathcal{B}(L_\infty(\mathbb{T}^\infty))$. Multipliers of $\mathcal{H}_2^* = \mathcal{H}_\infty$

Proposition: Dirichlet polynomials over the rationals are in \mathcal{T}_p

For $Q = \sum_{n=1}^N a_{q_n} q_n^{-s}$, the Toeplitz operator $T_Q : \mathcal{H}_p \rightarrow \mathcal{H}_p$ is bounded.

Theorem

$\mathcal{T}_p \subset \mathcal{B}(L_\infty(\mathbb{T}^\infty))$ and $\|\mathcal{B}^{-1}(D)\|_{L_\infty(\mathbb{T}^\infty)} \leq \|T_D\|$.

- Approximation of T_D by truncations of $T_{\mathcal{B}^{-1}(D)}$ as operator on $H_p(\mathbb{D}^N)$
- properties of spectrum of finite dimensional Toeplitz operators.

Corollary

Set of multipliers \mathcal{H}_p^* is contained in \mathcal{H}_∞ .

$\mathcal{A}(\mathbb{C}_+) =$ algebra of Dirichlet series uniformly continuous on $\overline{\mathbb{C}_+}$.
 $\mathcal{A}(\mathbb{C}_+) \subset \mathcal{H}_\infty$

Theorem

*For $p \neq 2$, there exists a co-analytic Dirichlet series $D \in {}_J\mathcal{A}(\mathbb{C}_+)$ such that $D \notin \mathcal{T}_p$ i.e. T_D is not a bounded operator.
In particular, $\mathcal{T}_p \subsetneq \mathcal{B}(L_\infty(\mathbb{T}^\infty))$*

Corollary

For $p \neq 2$, $\mathcal{M}(\mathcal{H}_p^) \subsetneq {}_J\mathcal{A}(\mathbb{C}_+) \subsetneq {}_J\mathcal{H}_\infty$.*

Results in dynamics: adjoint of multipliers

Theorem (Godefroy-Shapiro Type)

For $D \in \mathcal{H}_\infty$, M_D^ is hypercyclic (and chaotic) iff $D(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$.*

Theorem (Zero-One Law)

If M_D^ has an orbit with nonzero limit point, then it is hypercyclic.*

Theorem (Trichotomy)

For D with $\sigma_u(D) < 0$, exactly one holds:

- *M_D^* is hypercyclic*
- *All orbits $\rightarrow 0$*
- *All nonzero orbits $\rightarrow \infty$*

Results in dynamics: Toeplitz Operators

Theorem

Godefroy-Shapiro type For co-analytic $D \in \mathcal{T}_p$, T_D is hypercyclic iff $jD(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$.

Theorem (Zero-One Law)

For co-analytic $D \in \mathcal{T}_p$, if T_D has an orbit with nonzero limit point, then it is hypercyclic.

- M_{jD}^* is quasiconjugated to T_D through inclusion $\mathcal{H}_p \hookrightarrow \mathcal{H}_q^*$
- Finite-dimensional restrictions determine dynamics

Open Questions

- Explicit description of \mathcal{T}_ρ (symbols of bounded Toeplitz operators)
- Characterization multipliers of \mathcal{H}_ρ^*
- Does the trichotomy for orbits of T_D hold? \Leftarrow it does for co-analytic Dirichlet polynomials

Thank You!