Stochastic optimization in a jump-diffusion model: a singular stochastic control problem leading to a free boundary problem.

Nora Muler

Universidad Torcuato Di Tella,

Coauthors: Hansjörg Albrecher (Lausanne University) and Pablo Azcue

Dedicado a nuestro querido amigo y co-autor Pablo…

Outline

- Dividend problem in insurance
    - Rich source for stochastic control problems
    - Non-standard problem formulations

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  - Rich source for stochastic control problems
  - Non-standard problem formulations
- Shot-noise Cox process for claim arrivals

  ↪ Motivation from **Catastrophe Insurance**
- Solve two-dimensional control problem
- Numerical Illustrations / Interpretations

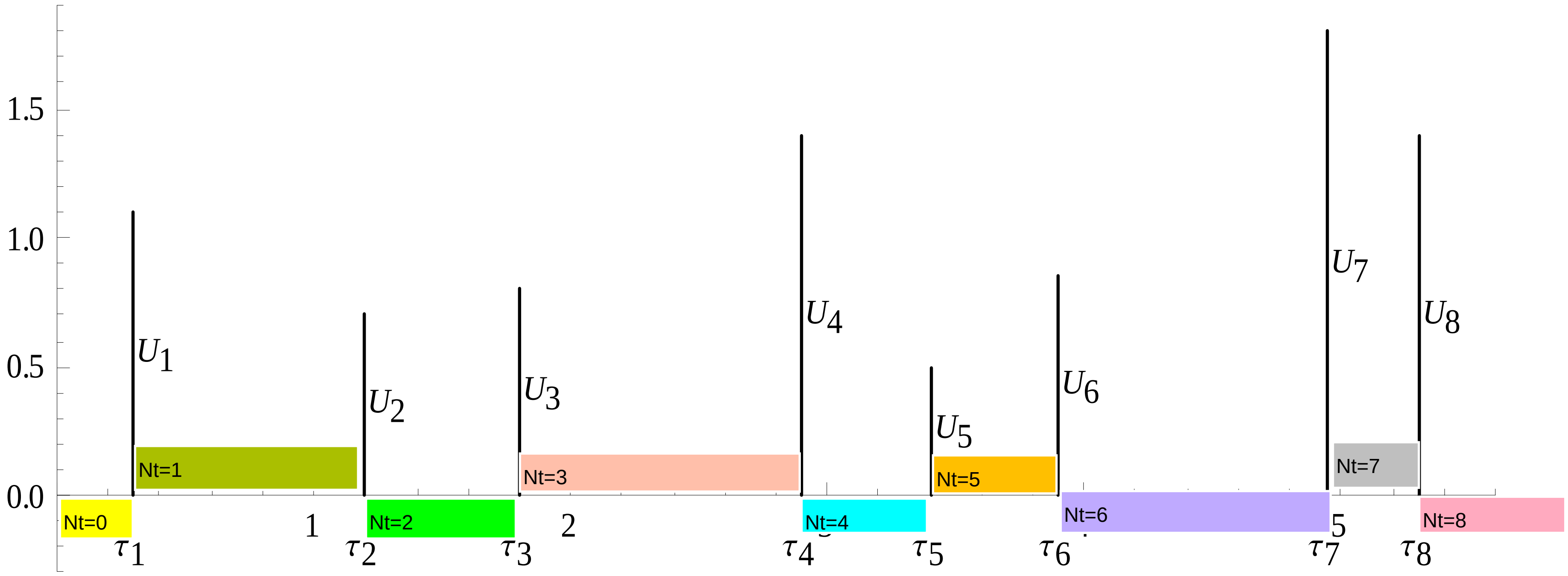
THE MODEL

Compound process with constant premium:

The insurance company receives a steady stream of money from its policyholders pdt (the insured) and pays out claims from its insured clients.

- $\tau_1 < \tau_2 < \tau_3 < \cdots$ are the times of the claims, ordered by their arrival.
- $U_1, U_2, U_3, \ldots$ are the positive sizes of the corresponding claims.
- N_t is a process that counts the number of claims the company has received up to time t, that is:

$$N_t = \max\{i : \tau_i \leq t\}.$$


1.5
1.0
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1
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5
U_1
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U_3
U_4
U_5
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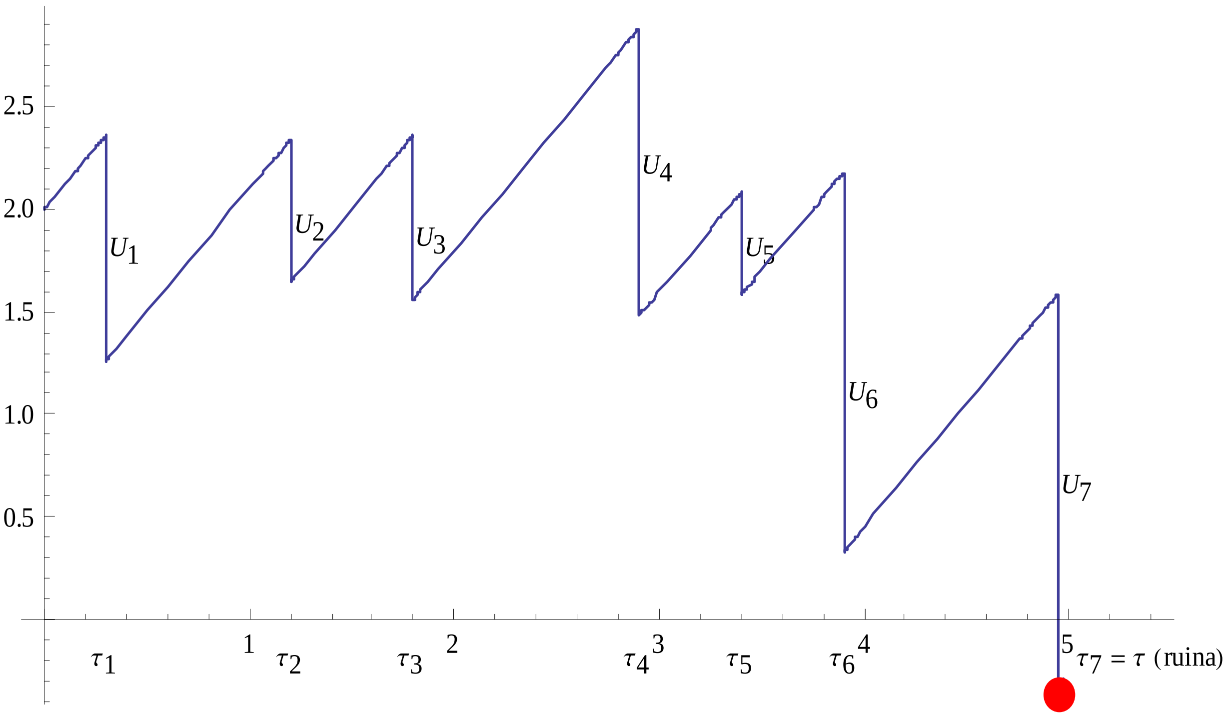
U_8
τ_1
τ_2
τ_3
τ_4
τ_5
τ_6
τ_7
τ_8
Nt=0
Nt=1
Nt=2
Nt=3
Nt=4
Nt=5
Nt=6
Nt=7
Nt=8

So, the free surplus (or reserve) of the insurance company is given by,

$$X_t = x + pt - \sum_{k=1}^{N_t} U_k$$

- x is the initial surplus, $p > 0$ is the premium rate that receives the company continuously from the clients.
- τ_j is the time of arrival of the j-th claim and $N_t = \#\{j : \tau_j \leq t\}$ is the counting process of arrivals of claims up to time t. We call τ the ruin time that is the first time in which the surplus becomes negative.
- U_k corresponds to the size of the $k-th$ claim and $(U_k)_{k\geq 1}$ are non-negative i.i.d. random variables with distribution function F_u independent of N_t.



Simulated reserve process X(t) until ruin

The ruin time is the time in which the reserve becomes negative

In the (classical) Compound Poisson case :

- $P(\tau_{i+1} - \tau_i \leq t) = 1 - e^{-\lambda t}$, and therefore the waiting time between one claim and the next also follows an exponential distribution with parameter λ. The variable $\tau_{i+1} - \tau_i$ is independent of $\tau_{k+1} - \tau_k$ with $k < i$, meaning the process is "memoryless".

N_t is called a Poisson process with rate λ and for each fixed t , it follows a Poisson distribution with parameter λt :

$$P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, k = 0, 1, 2, \ldots.$$

Also $E(N_t - N_s) = \lambda(t - s), t > s$ (that is, the expected number of claims in any time interval is λ× the length of the time interval).

Natural catastrophes: Severe convective storms

Simulated tornado-producing supercell (30m resolution, EF5 near El Reno and Oklahoma City, 2011)

Orf et al. (2017)

Instead of Poisson...

MOTIVATION FOR THE COX PROCESS

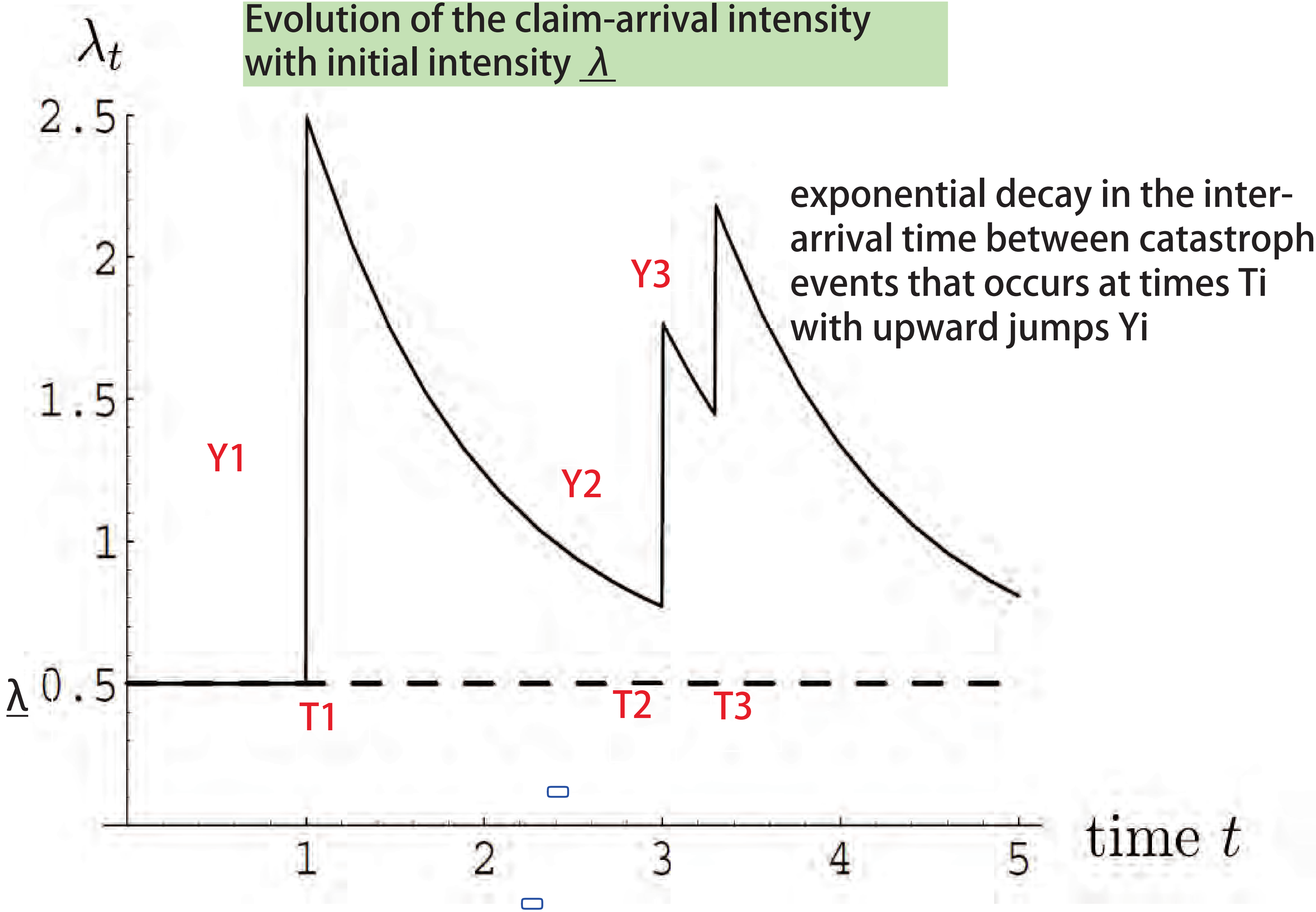
- **For catastrophic events, the assumption that resulting claims occur in terms of an homogeneous Poisson process is inadequate.**

- **The shot noise process measure the number of claims due to catastrophic event. It is taken into account the frequency, magnitude and inter-arrival times of the catastrophes (see for instance Dassios and Jang (2003), Albrecher and Asmussen (2005), Dassios, Jang and Zhao, (2015) and Pojer and Thonhauser (2023)).**

- **The catastrophic events have the effect of sudden increases of the claim intensity. As time passes, the shot noise process decreases until another catastrophic event occurs which will result in a positive jump.**

- **We assume here that the claim arrival process is a superposition of an homogeneous Poisson process (due to "normal" events) and a Cox process (due to catastrophic events).**

Evolution of the claim-arrival intensity with initial intensity $\underline{\lambda}$



exponential decay in the inter-arrival time between catastrophic events that occurs at times Ti with upward jumps Yi

Shot noise Intensity Process

We assume that the claim intensity λ_t is given by

$$\lambda_t = \underline{\lambda} + e^{-dt}(\lambda - \underline{\lambda}) + \sum_{k=1}^{\widetilde{N}_t} Y_k \, e^{-d(t-T_k)}$$

- $\lambda \geq \underline{\lambda}$ is the initial intensity and if the initial intensity is $\lambda > \underline{\lambda}$, then $\lambda_t > \underline{\lambda}$ for all $t \geq 0$. Here the intensity is not constant anymore
- $\underline{\lambda}$ is the intensity of the homogeneous Compound Poisson process.
- $d > 0$ is the exponential decay.
- Y_k are i.i.d. positive random variables with distribution function F_y and finite mean, and corresponds to the jump size of the catastrophe k.
- $\widetilde{N}_t = \#\{k : T_k \leq t\}$ is an homogeneous Poisson process of constant intensity β that counts the number of catastrophes up to time t, T_k is the arrival times of the catastrophe k.

Counting claim Process Nt: an inhomogeneous Poisson Process

Inhomogeneous vs Homogeneous:

$$\Lambda_t = \int_0^t \lambda_s ds \Rightarrow \lambda t.$$

$$\mathrm{P}(N_t = k) = \mathrm{E}\left(e^{-\Lambda_t}\frac{1}{k!}(\Lambda_t)^k\right) \Rightarrow e^{-\lambda t}\frac{1}{k!}(\lambda t)^k$$

$$\mathrm{P}(\tau_1 > t) = \mathrm{E}[e^{-\Lambda(t)}] \Rightarrow e^{-\lambda t}$$

The Controls

Dividend strategies

The company uses part of the surplus to pay dividends to the shareholders .

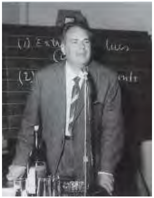
- The dividend payment strategy L_t is the total amount of dividends paid by the company up to time t. The dividend strategy should be non-decreasing, càdlàg (right continuous with left limits), predictable, and the company is not allowed to pay more dividends than their current surplus.
- The controlled surplus process can be written as

$$X_t^L = X_t - L_t.$$

- The ruin time of the controlled process depends also on the dividend strategy and corresponds to the first time in which the controlled surplus becomes negative.

Optimal Dividend Criterion

Proposed by:



B. de Finetti (1957)

Expected sum of discounted dividend payments until ruin

$$\mathbb{E}_x\Big(\int_0^{\tau} e^{-qt}\, dL_t\Big) \longrightarrow \text{maximize!}$$

Stochastic Control Problem:

- Compound Poisson: e.g. Gerber (1969), Schmidli (2008), Azcue & M. (2014)
- Diffusion: e.g. Gerber (1972), Jeanblanc-Piqué & Shiryaev (1994), Asmussen & Taksar (1997)

e.g. Albrecher & Thonhauser (2009), Avanzi (2009)

Both cases: 1-D

The compound Cox process generated by a shot-noise intensity makes this problem two-dimensional

Why?

Because the claim intensity is changing with time, **so it is necessary to keep track not only on the current surplus but also on the current intensity**.

More precisely

Let us define the set of admissible strategies $\Pi_{x,\lambda}$ where x is the initial surplus and λ is the initial intensity of the claim-arrival process N_t.

The expected discounted dividends until ruin for an admissible strategy $L = (L_t)_{t\geq 0} \in \Pi_{x,\lambda}$ is given by:

$$J(L) = E\left[\int_0^\tau e^{-qs} dL_s\right]$$

where $q > 0$ is the discounted parameter.

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The optimal value function is a two-dimensional problem that depends on the initial surplus and initial intensity of the arrival claims and is given by,

$$V(x,\lambda) = \sup_{L\in\Pi_{x,\lambda}} J(L).$$

Basic Properties

- The optimal value function V is non-decreasing on x (more initial surplus implies a better value function).
- It is non-increasing on λ (the number of claim arrivals is larger as λ increases and so the ruin occurs before).
- The asymptotic condition on λ is given by $lim_{\lambda\to\infty}V(x,\lambda) = x.$

  (Because the intensity of claims go to infinity)
- V has a linear growth on x at infinity because

$$V(x,\lambda) \leq \overbrace{x}^{\text{take the money}} + \overbrace{\int_0^\infty pe^{-qs}ds}^{\text{pay for ever the incoming premium}} = x + p/q$$

Hamilton-Jacobi-Bellman equation

The (HJB) equation is given by

$$\max\{\mathrm{L}(V)(x,\lambda), 1 - V_x(x,\lambda)\} \;=\; 0,\; x \geq 0, \underline{\lambda} \leq \lambda,$$

where $L(V)(x,\lambda) = L_1(V)(x,\lambda) + L_2(V)(x,\lambda)$.

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1.

$$L_1(V)(x,\lambda) = \overbrace{-qV(x,\lambda)}^{\text{discounted term}} + pV_x(x,\lambda) - \lambda V(x,\lambda) + \lambda \int_0^x V(x-\alpha,\lambda)\,dF_u(\alpha)$$

2.

$$L_2(V)(x,\lambda) = -d(\lambda - \underline{\lambda})V_\lambda(x,\lambda) - \beta V(x,\lambda) + \beta \overbrace{\int_0^\infty V(x,\lambda+\gamma)\,dF_Y(\gamma)}^{\text{upward jumps on the intensity}}$$

Characterization Theorem

The optimal value function V is the smallest viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation

$$\max\{\mathrm{L}(V)(x,\lambda), 1 - V_x(x,\lambda)\} = 0,\ x \geq 0, \underline{\lambda} \leq \lambda,$$

satifying the two boundary conditions:

(a) Linear growth at infinity on the variable x.

(b)$lim_{\lambda\to\infty} V(x,\lambda) = x$.

Verification Theorem (our way to identify the good solution)

1. Consider $(L_{x,\lambda})_{x\geq 0,\lambda\geq\underline{\lambda}}$ where each $L_{x,\lambda}$ is an admissible strategy for initial surplus x and initial intensity λ. If the function $W(x,\lambda) = J(L_{x,\lambda})$ is a viscosity solution (or supersolution) of the HJB equation for all $x \geq 0, \lambda \geq \underline{\lambda}$, then W is the optimal value function.

Note that by the HJB equation,

$$\max\{\mathrm{L}(V)(x,\lambda), 1 - V_x(x,\lambda)\} \; = \; 0, \, x \geq 0, \underline{\lambda} \leq \lambda,$$

both $L(V) \leq 0$ **and** $V_x \geq 1$ **and at each** (x,λ) **one of the operators should be zero**.

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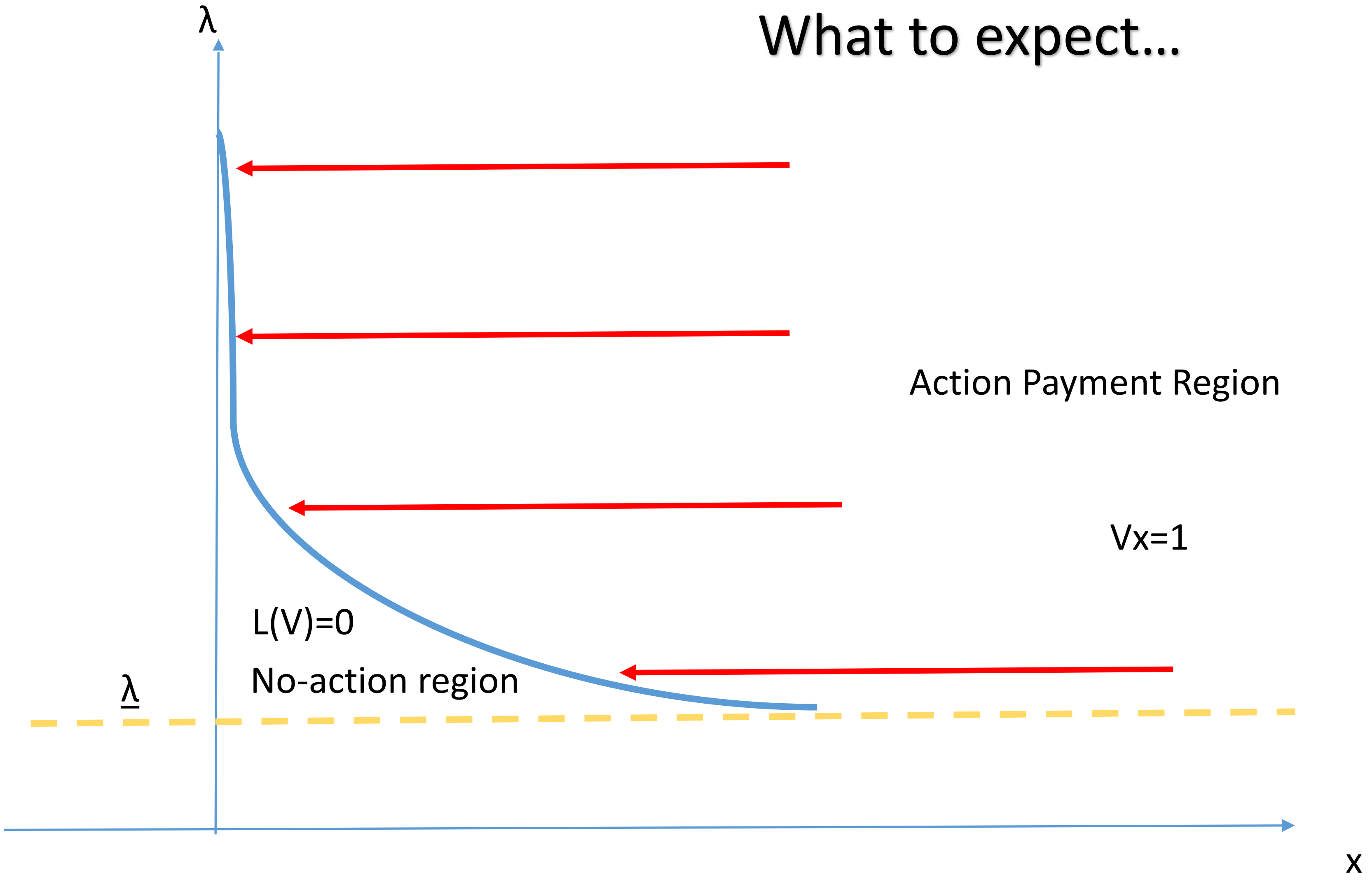
both $L(V) \leq 0$ and $V_x \geq 1$ and at each (x,λ) one of the operators should be zero.

So. the HJB equation suggests that the optimal strategy would depend on the current controlled surplus X_t^L and on the current intensity process λ_t in the following way,

1. If (X_t^L, λ_t) is in the interior of the set where $L(V) = 0$ the optimal strategy consists on not paying dividends (non-change region).
2. If (X_t^L, λ_t) is in the interior of the set where $V_x = 1$ the company should pay a lump sum as dividends (change region).
3. We have "free boundaries" between the non-change region and the change region .

Note that the optimal strategy would depend on the current controlled surplus X$_t^L$ and on the current intensity process λ_t . Hence, the optimal strategy is Markovian on the two-dimensional state space $[0,\infty) \times [\underline{\lambda},\infty)$ where $(x,\lambda)\in [0,\infty) \times [\underline{\lambda},\infty)$.

What to expect…



Numerical Approximation: admissible strategies defined in suitable grids

It is a challenging problem to find numerically the curve (**or curves**?) **that split the no-action and the action region**.

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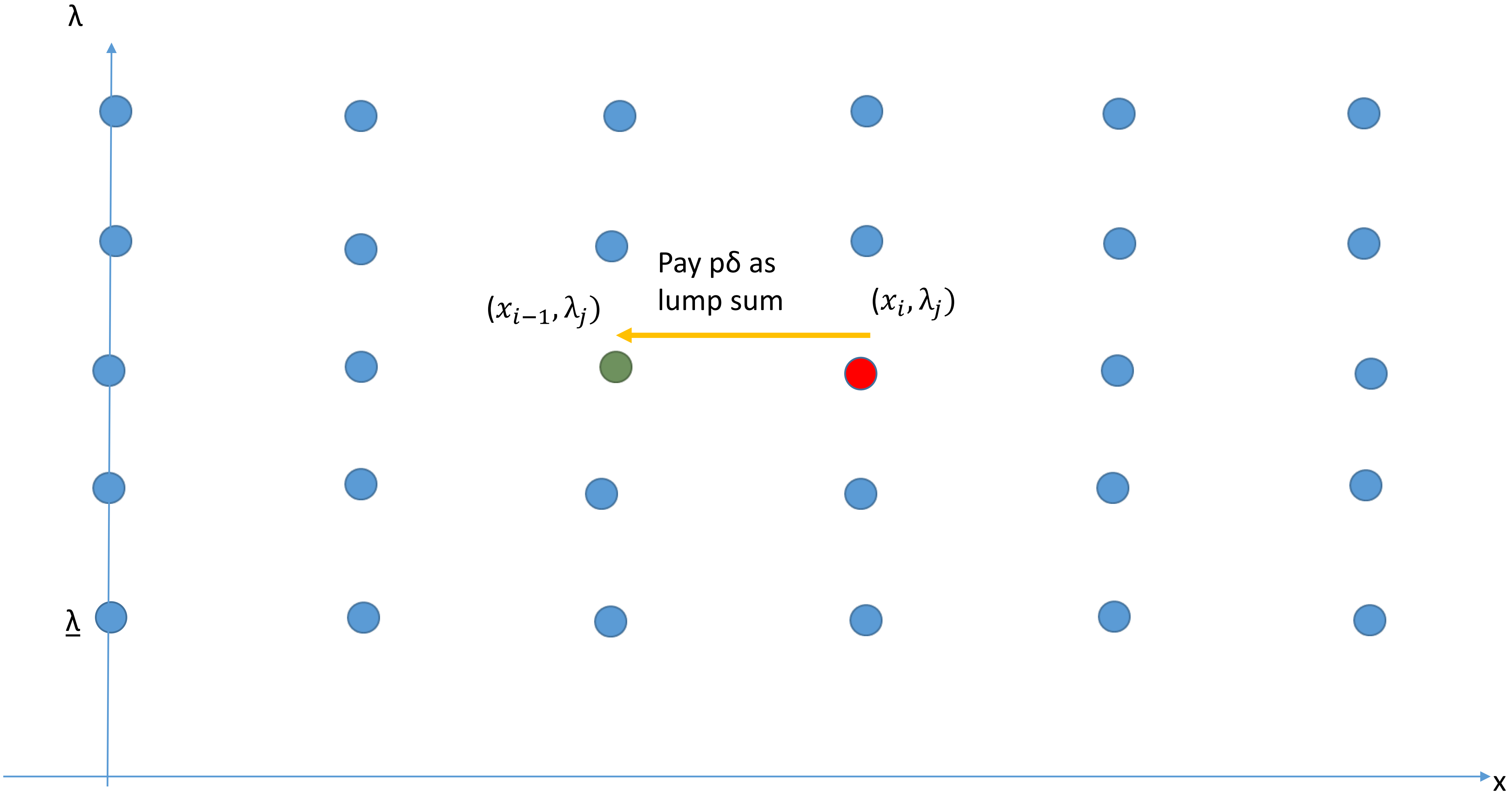
We have developed a convergent numerical algorithm. It is based on local strategies in a grid: the idea is to consider strategies that either not pay dividends (non-action region) or pay a lump sum (maybe small) as dividends in such a way that after paying dividends, the current surplus is in the grid.

The local controls E at each point of the grid (x_i, λ_j) are the following:

- Action-local strategy: The company pays immediately $p\delta$ as dividends so the point in the grid becomes (x_{i-1}, λ_j) .

LOCAL STRATEGY: pay immediately dividends



(x_{i-1}, λ_j)

Pay pδ as lump sum

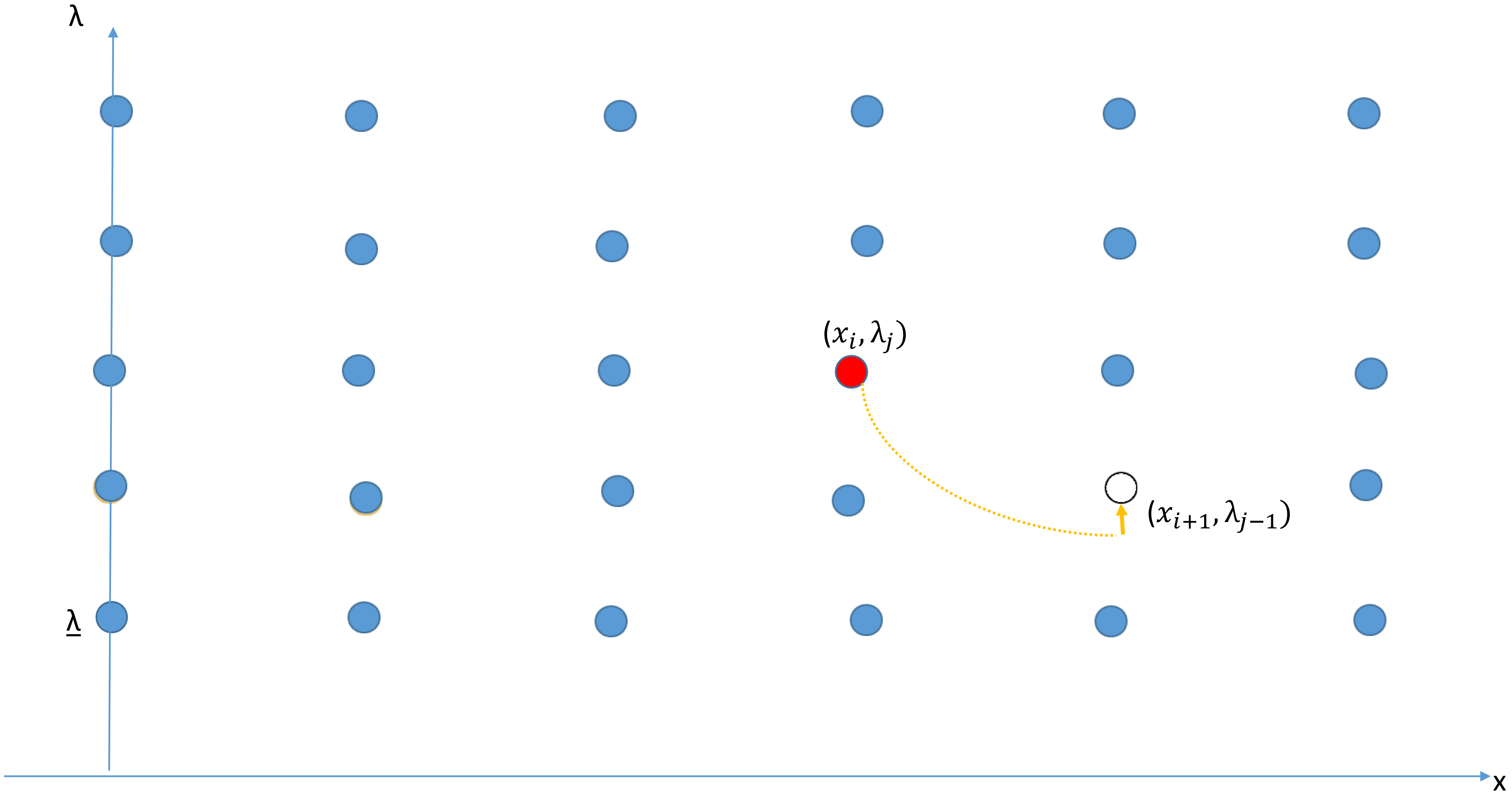
(x_i, λ_j)

The local controls E at each point of the grid (x_i, λ_j) are the following:

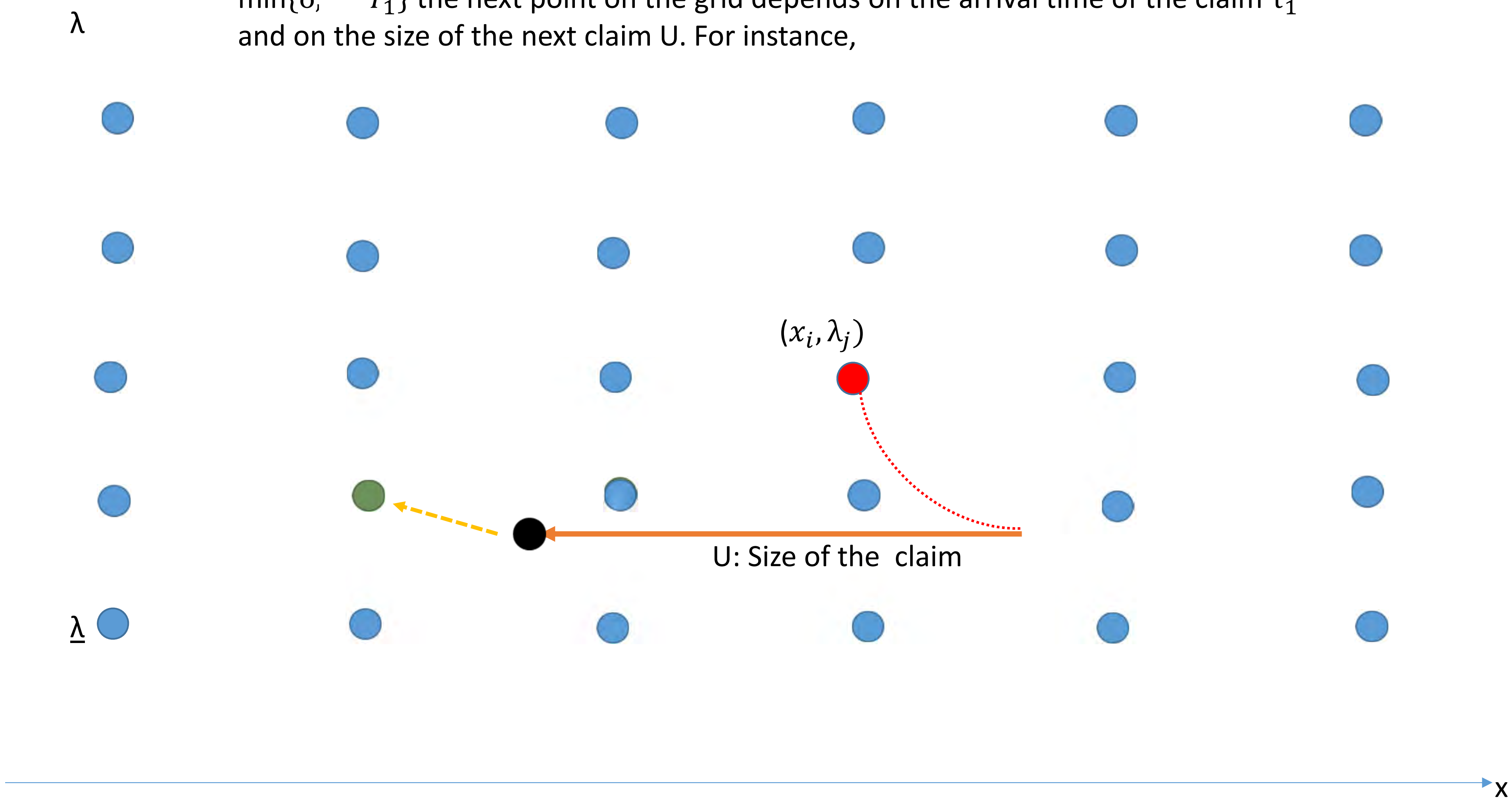
- Action-local strategy : The company pays immediately $p\delta$ as dividends so the point in the grid becomes (x_{i-1}, λ_j) .

- Non-action local strategy : The company does not pay dividends until the minimum between the time in which the surplus reaches the point $x_{i+1} = x_i + p\delta$ (that means $\Delta t = \delta$), the time of the arrival of the next claim or the time that an upward jump on the intensity process occurs.

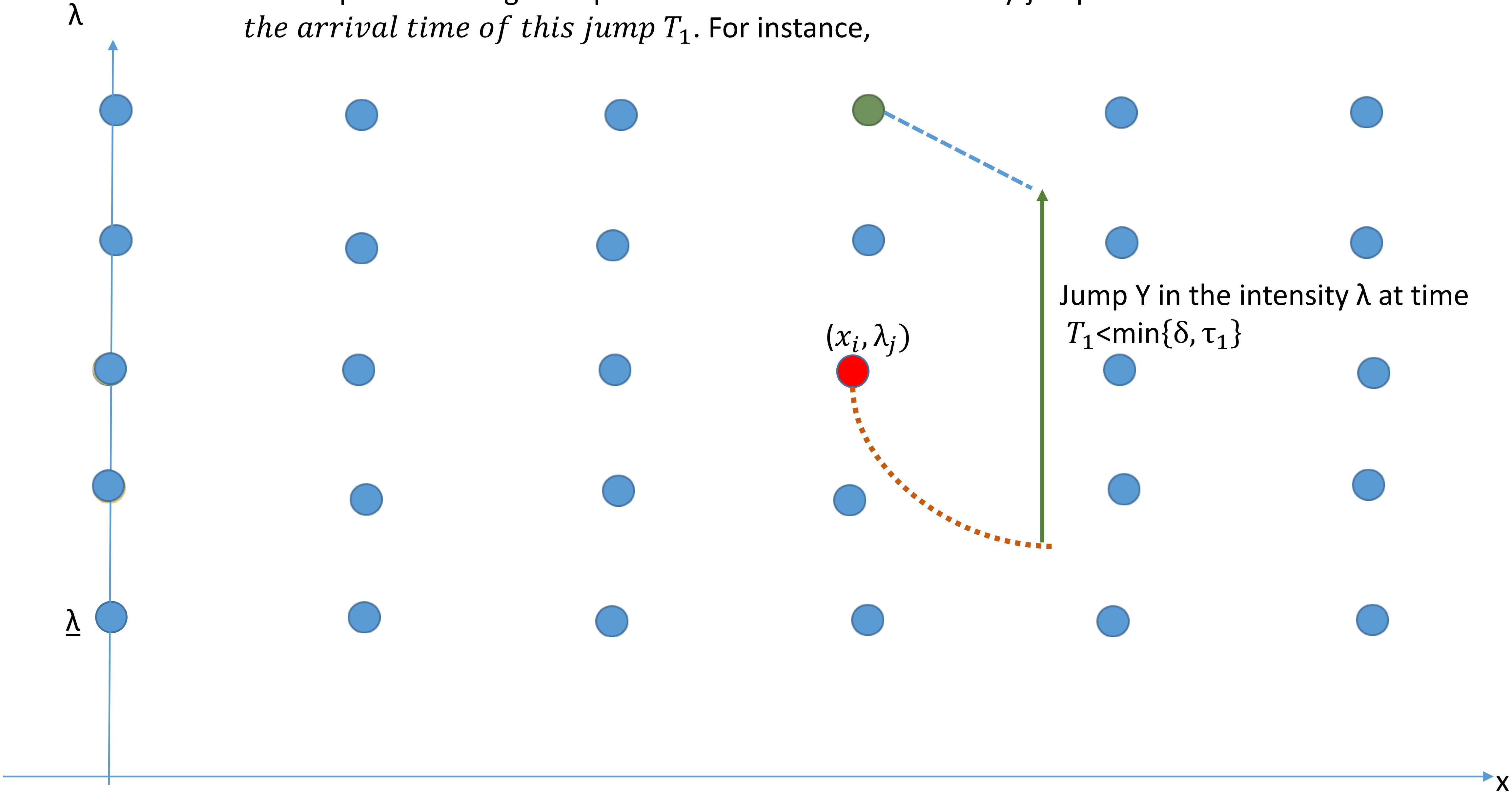
LOCAL STRATEGY (NO-ACTION): not to pay dividends until min$\{\delta, \tau_1, T_1\}$: If $\delta<\min\{\tau_1, T_1\}$



LOCAL STRATEGY (NO-ACTION) not to pay dividends until $\min\{\delta, \tau_1, T_1\}$: If $\tau_1 < \min\{\delta, T_1\}$ the next point on the grid depends on the arrival time of the claim τ_1 and on the size of the next claim U. For instance,



LOCAL STRATEGY (NO-ACTION): not to pay dividends until $\min\{\delta, \tau_1, T_1\}$: If $T_1<\min\{\delta, \tau_1\}$, next point on the grid depends on the size of the intensity-jump Y and $the\ arrival\ time\ of\ this\ jump\ T_1$. For instance,



Jump Y in the intensity λ at time $T_1<\min\{\delta, \tau_1\}$

We define $V^{\delta,\triangle}$ as the "optimal value function" between all the possible combination of the two local controls in the grid until ruin. More precisely, for any (x_i, λ_j)

$$V^{\delta,\triangle}(x_i, \lambda_j) = \sup_{L \in \Pi^{\delta,\Delta}_{x_i,\lambda_j}} V_L(x_i, \lambda_j),$$

We define $V^{\delta,\triangle}$ as the "optimal value function" between all the possible combination of the two local controls in the grid until ruin. More precisely, for any $(x_i,\lambda_j) \in \mathrm{G}^{\delta,\Delta}$:

$$V^{\delta,\triangle}(x_i,\lambda_j) = \sup_{L\in\Pi^{\delta,\Delta}_{x_i,\lambda_j}} V_L(x_i,\lambda_j),$$

Theorem

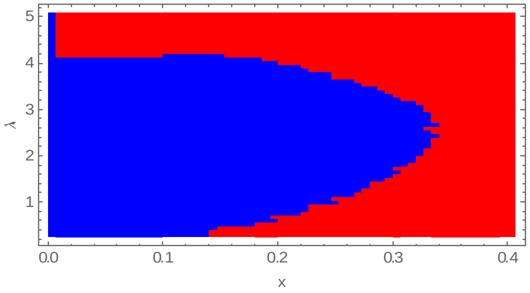
$V^{\delta_k,\triangle_k} \nearrow V$ uniformly in compact sets as $k \to \infty$.

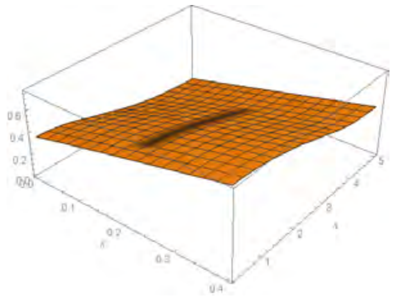
We prove this result using that a viscosity supersolution of the HJB equation that is a limit of admissible strategies is the optimal value function (Verification Result).

Examples

Exponential distributions for the claim-sizes and for the size of catastrophes:

$\underline{\lambda} = 1/4$, $F_Y(x) = 1 - e^{-\frac{1}{2}x}$ (dist. size of catastrophes), $\beta = 1/2$ (⇒so we expect a catastrophe to happen in mean every 2 years), $d = 7/10$ (exponential decay to $\underline{\lambda}$), $F_U(x) = 1 - e^{-\frac{1}{10}x}$(dist. of the claim-sizes), $q = 1/5$ (discounted parameter) and $\eta = 1/5$ (safety loading).





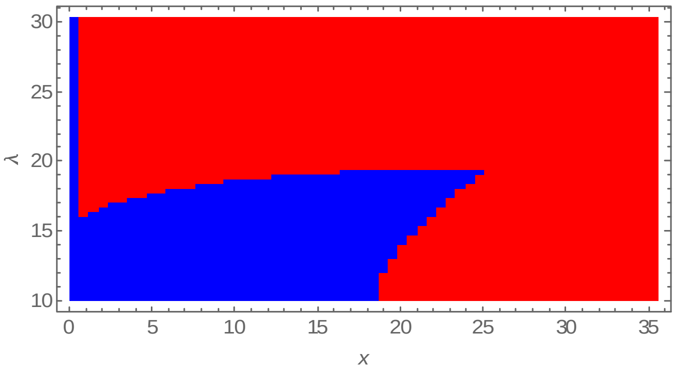
Optimal Strategy

Optimal value function

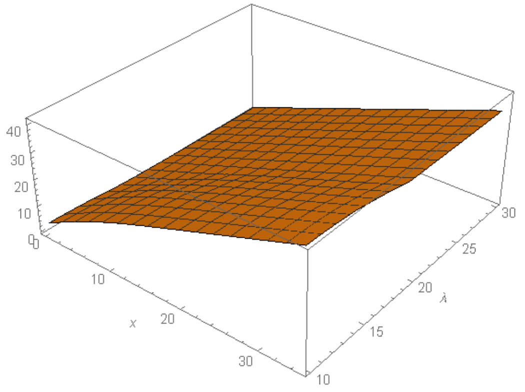
Blue: Non-action region, Red: Action region

Gamma distributions for the claim-sizes and exponential for the size of catastrophes:

$\underline{\lambda} = 10$, $F_Y(x) = 1 - e^{-\frac{1}{2}x}$ (dist. size of catastrophes), $\beta = 1/5$ (frequency of catastrophes is a random exponential with mean 5), $d = 1/5$ (the exponential decay to $\underline{\lambda}$), $F_U(x) = 1 - (1 + x)e^{-x}$ (size of claims), $q = 1/10$ (discounted parameter) and $\eta = 7/100$ (safety loading)



Optimal Strategy

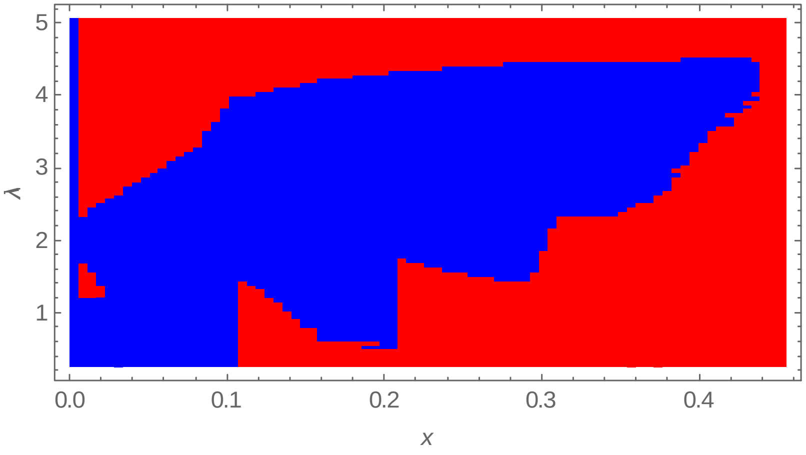


Optimal Value Function

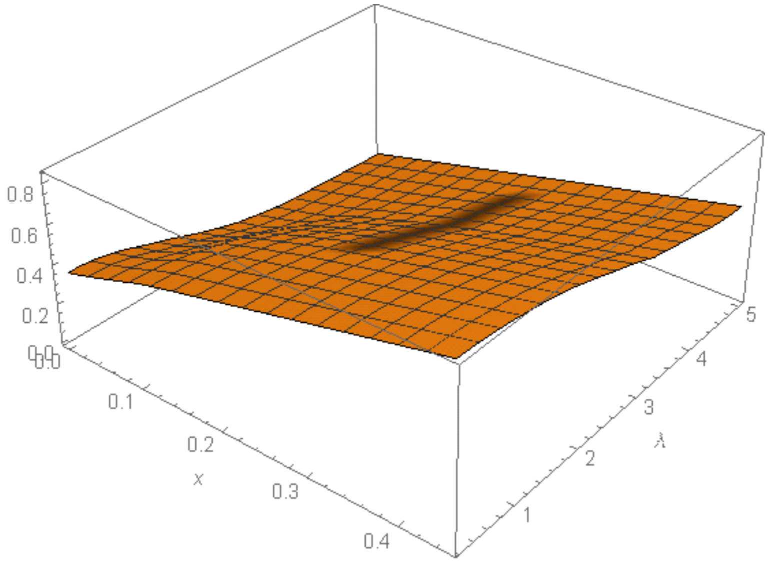
Blue: Non-action region,
Red: Action region

Fixed claim-sizes and exponential distribution for the size of catastrophes:

$\underline{\lambda} = 1/4$, $F_Y(x) = 1 - e^{-\frac{1}{2}x}$ (dist. size of catastrophes), $\beta = 1/2$ (⇒frequency of catastrophes has mean 2), $d = 7/10$ (exponential decay to $\underline{\lambda}$), $F_U(x) = I_{x\geq 1/10}$ (fixed claim-sizes), $q = 1/10$ (discounted parameter) and $\eta = 7/100$



Optimal Strategy



Optimal Value Function

Blue: Non-action region, Red: Action region

Possible Extensions

- Irreversible climate change (we are working on this).
- Cyclic climate changes like "la niña"/"el niño".
- Self-excited process in the intensity like the Hawkes process (cyber risk, contagious models).

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MUCHAS GRACIAS!!

Additional Material

Definitions of Viscosity solutions

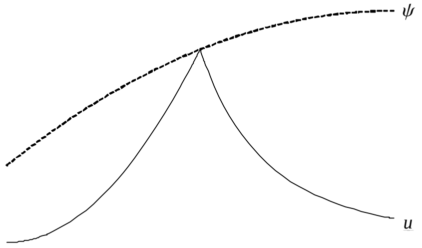
A function $\underline{u} : J \to \mathbf{R}$ is a viscosity subsolution at $x \in J$ if it is locally Lipschitz and any continuously differentiable function $\psi : J \to \mathbf{R}$ with $\psi(x) = \underline{u}(x)$ and such that $\underline{u} - \psi$ reaches the maximum at x satisfies

$$L(x, \psi(x), \psi'(x), \psi) \geq 0.$$

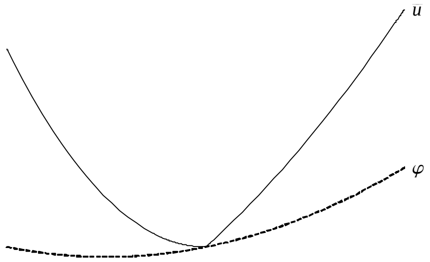
A function $\bar{u} : J \to \mathbf{R}$ is a viscosity supersolution at $x \in J$ if it is locally Lipschitz and any continuously differentiable function $\varphi : J \to \mathbf{R}$ with $\psi(x) = \bar{u}(x)$ and such that $\bar{u} - \varphi$ reaches the minimum at x satisfies

$$L(x, \varphi(x), \varphi'(x), \varphi) \leq 0.$$

If a function $u : J \to \mathbf{R}$ is both a subsolution and a supersolution at $x \in J$ it is called a viscosity solution at x.

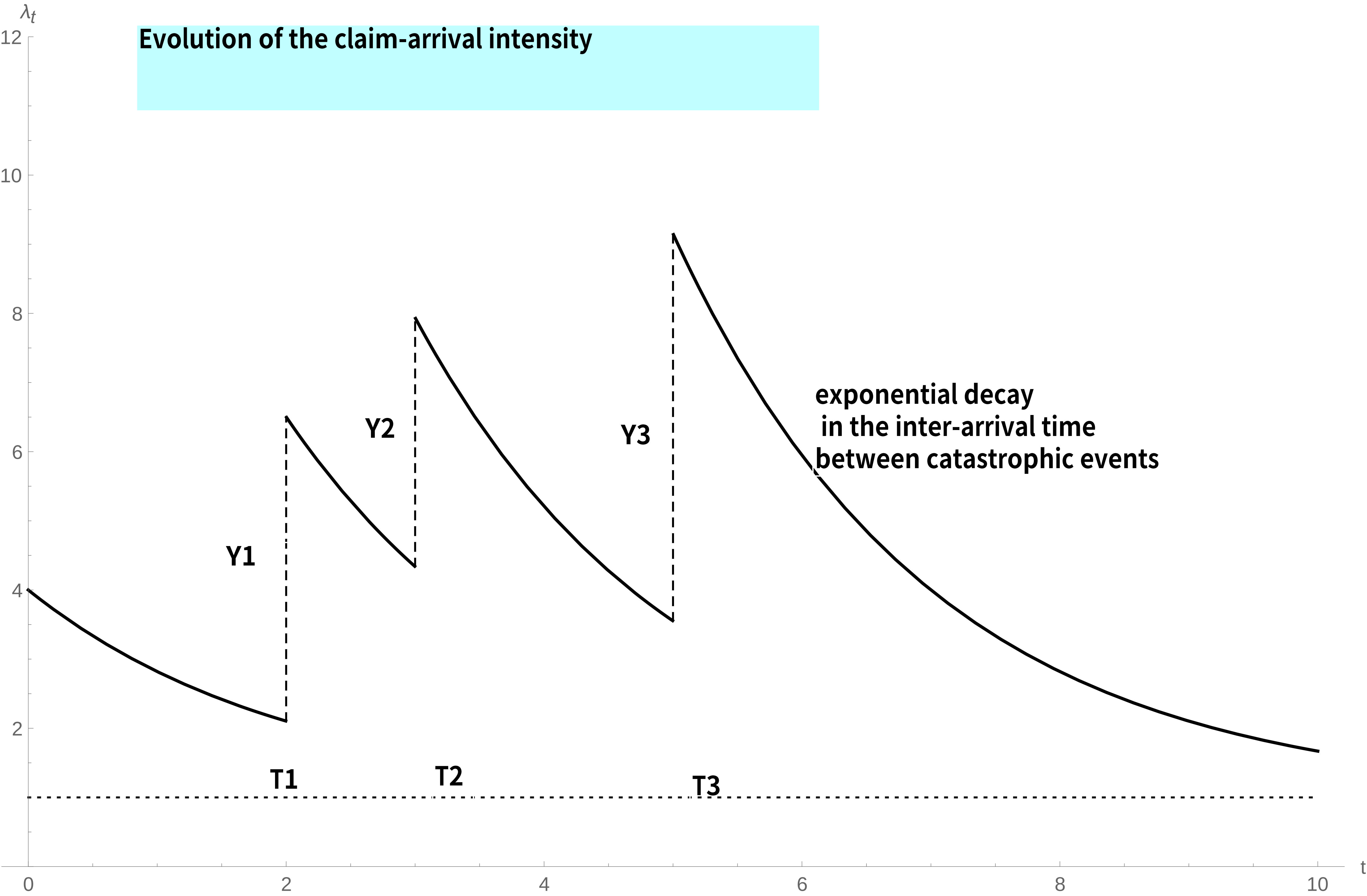


(a)



(b)

As we show in Figures (a) and (b), the test function ψ touches $\underline{u}$ from above and the test function φ touches $\bar{u}$ from below, their derivatives $\psi'(x)$ and $\varphi'(x)$ correspond to the super- and sub-differentials at x respectively.


λ_t
Evolution of the claim-arrival intensity
12
10
8
6
4
2
Y1
Y2
Y3
exponential decay
in the inter-arrival time
between catastrophic events
T1
T2
T3
0
2
4
6
8
10
t

Furthermore, we assume that the premium rate p is obtained using the expected value principle of the asymptotic distribution of λ_t with a positive safety loading η. That is,

$$p = (1+\eta)E(U_1)\lim_{t\to\infty}\mathrm{E}\left(\frac{\Lambda_t}{t}\right).$$

where

$$\lim_{t\to\infty}\mathrm{E}\left(\frac{\Lambda_t}{t}\right) = \lim_{t\to\infty}\mathrm{E}\left(\frac{1}{t}\int_0^t \lambda_s ds\right) = \underline{\lambda} + \overbrace{\frac{\beta\,\mathrm{E}(Y_1)}{d}}^{\text{Shot-Noise term}}$$

Characterization Theorem

The optimal value function V is the smallest viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation

$$\max\{\mathrm{L}(V)(x,\lambda), 1 - V_x(x,\lambda)\} = 0,\ x \geq 0, \underline{\lambda} \leq \lambda,$$

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(a) Linear growth at infinity on the variable x.

(b)$lim_{\lambda \to \infty} V(x,\lambda) = x$.

From this result it can be proved: $V(x,\lambda) = x - p/q + V(p/q,\lambda)$

So: it is always optimal to pay a lump sum of dividends if the surplus is above p/q.

The infinitesimal generator

The **infinitesimal generator** G of a Markov process $\overline{S} = (S_t)_{t\geq 0}$ with $S_0 = x$ is the operator defined on the continuously differentiable functions by

$$\mathrm{G}(\overline{S}, f)(x) = \lim_{t\to 0} \frac{E_x(f(S_t)) - f(x)}{t}.$$

In the (classical) Compound Poisson case :

- $P(\tau_{i+1} - \tau_i \leq t) = 1 - e^{-\lambda t}$, and therefore the waiting time between one claim and the next also follows an exponential distribution with parameter λ. The variable $\tau_{i+1} - \tau_i$ is independent of $\tau_{k+1} - \tau_k$ with $k < i$, meaning the process is "memoryless".

because...
the exponental distribution has the key propery that the probability of an event occurring in the future is independent of how long it has already been waiting for the event.