A two-player zero-sum probabilistic game that approximates the mean curvature flow

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Contents

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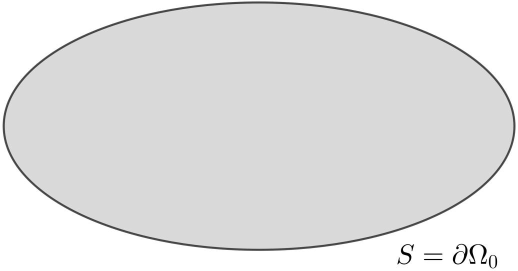

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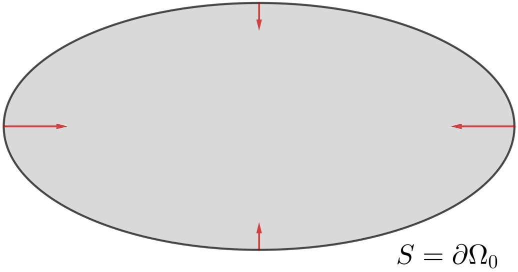

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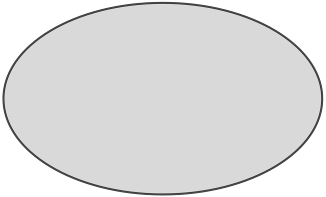

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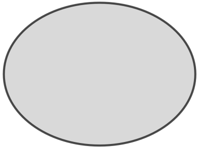

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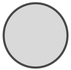

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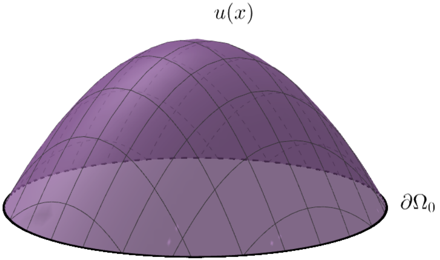
The movement by mean curvature of a hypersurface and its associated elliptic equation

Let us start with a hypersurface $S = \partial\Omega_0 \subset \mathbb{R}^N$, that is the boundary of a connected and strictly convex domain. We will use a level set approach to describe this geometric evolution.

$$\Omega_t = \{x : u(x) > t\}, \quad t \geq 0.$$



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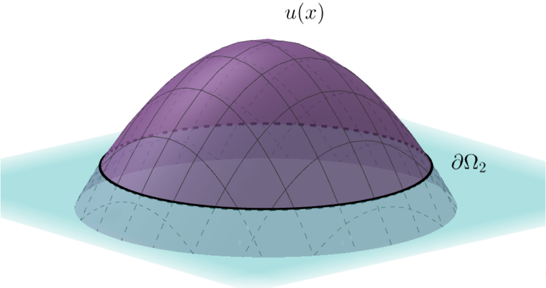

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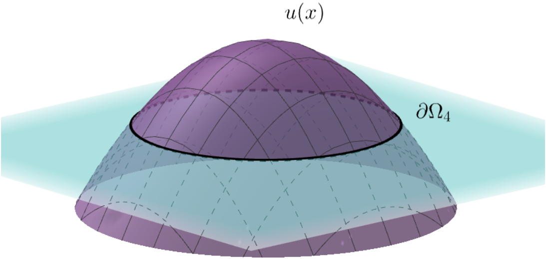

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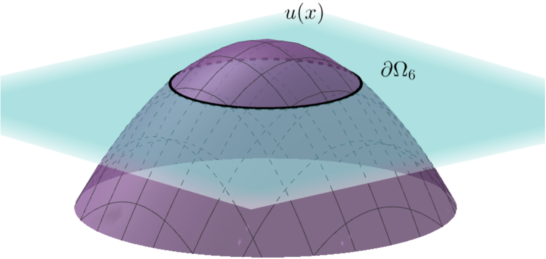

Take $x \in \partial\Omega_t$ a regular point ($\nabla u(x) \neq 0$). We have $\nabla u(x) \perp \partial\Omega_t$ and for a unitary vector $v \perp \nabla u(x)$ (v is tangential to the hypersurface $\partial\Omega_t$) the quantity $-\langle D^2 u(x) v, v\rangle$ gives the curvature of $\partial\Omega_t$ in the direction of v.

The mean of the principal curvatures of $\partial\Omega_t$ is given by

$$\kappa = \sum_i \kappa_i = \operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big)(x) = \frac{1}{|\nabla u(x)|}\Big(\Delta u(x) - \Big\langle D^2 u(x)\frac{\nabla u}{|\nabla u|}(x), \frac{\nabla u}{|\nabla u|}(x)\Big\rangle\Big).$$

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The elliptic mean curvature equation:

$$\begin{cases} |\nabla u(x)|\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)(x) = -1, & x \in \Omega_0, \\ u(x) = 0, & x \in \partial\Omega_0. \end{cases}$$

or

$$\begin{cases} \Delta u(x) - \left\langle D^2 u(x) \frac{\nabla u}{|\nabla u|}(x), \frac{\nabla u}{|\nabla u|}(x) \right\rangle = -1, & x \in \Omega_0, \\ u(x) = 0, & x \in \partial\Omega_0. \end{cases}$$

Notice that when u solves the equation, then $v(x,t) = u(x) - t$ is a solution to

$$\frac{\partial v}{\partial t}(x,t) = \Delta v(x,t) - \left\langle D^2 v(x,t) \frac{\nabla v}{|\nabla v|}(x,t), \frac{\nabla v}{|\nabla v|}(x,t) \right\rangle,$$

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  The authors proved that there exists a unique viscosity solution and in addition a comparison principle holds.

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Two formulations

$$\begin{cases} \Delta u(x) - \left\langle D^2 u(x) \dfrac{\nabla u}{|\nabla u|}(x), \dfrac{\nabla u}{|\nabla u|}(x) \right\rangle = -1, & x \in \Omega_0, \\ u(x) = 0, & x \in \partial\Omega_0. \end{cases}$$

and

$$\begin{cases} \dfrac{1}{\mu(\{\langle \nabla u(x), v\rangle = 0\})} \displaystyle\int_{\langle \nabla u(x), v\rangle = 0} \dfrac{1}{2} \langle D^2 u(x) v, v\rangle d\mu(v) = -C, & x \in \Omega_0, \\ u(x) = 0, & x \in \partial\Omega_0. \end{cases}$$

with

$$C = \frac{1}{\mu(\langle e_N, v\rangle = 0)} \int_{\langle e_N, v\rangle = 0} \frac{1}{2} (v_1)^2 \, d\mu(v).$$

There are equivalent.

Contents

A probabilistic game approximation for the elliptic problem

The game is a probabilistic two-person zero-sum game.

- Paul (maximize)
- Carol (minimize)

Take $\varepsilon > 0$, $\Omega_0 \subset \mathbb{R}^N$ a **strictly convex** and bounded domain. Let us start in $x_0 \in \Omega_0$.

- Paul chooses a set of **unitary vectors** $A \subset S^{N-1}$ with $\sigma(A) \geq \frac{1}{2}\sigma(S^{N-1}) + \delta_\varepsilon$.
- Carol chooses a set $B \subset S^{N-1}$ with measure $\sigma(B) \geq \frac{1}{2}\sigma(S^{N-1}) + \delta_\varepsilon$.

Then, the next position of the game is given by

$$x_1 = x_0 + v_1 \varepsilon,$$

where the vector v_1 is randomly chosen (with uniform probability) in the set $A \cap B$.

The game ends when the position exits Ω_0 and Carol pays to Paul an amount

$$\varepsilon^2 K \times (\text{number of plays})$$

with K a constant that we will specify latter.

– The value of the game for **Paul** is ,

$$u_p^\varepsilon(x_0) = \inf_{S_c} \sup_{S_p} \mathbb{E}_{S_p,S_c}^{x_0}\left[\varepsilon^2 K \times (\text{number of plays})\right].$$

– The value for **Carol** is given by

$$u_c^\varepsilon(x_0) = \sup_{S_p} \inf_{S_c} \mathbb{E}_{S_p,S_c}^{x_0}\left[\varepsilon^2 K \times (\text{number of plays})\right].$$

We proved that the game has a value, that is

$$u^\varepsilon(x_0) := u_p^\varepsilon(x_0) = u_c^\varepsilon(x_0)$$

This function verifies the following **Dynamic Programming Principle (DPP)**

$$\begin{cases} u^\varepsilon(x) = \sup_A \inf_B \left\{ \fint_{A\cap B} u^\varepsilon(x + v\varepsilon)\, d\sigma(v) \right\} + \varepsilon^2 K, & x \in \Omega_0, \\ u^\varepsilon(x) = 0, & x \in \mathbb{R}^N \setminus \Omega_0. \end{cases}$$

In addition, the comparison principle holds. Finally, the value of the game is the unique solution to the DPP.

The value of the game converges to a function u.

We defined

$$\overline{u}(x) = \limsup_{\substack{\varepsilon\to 0^+\\ y\to x}} u^\varepsilon(y), \quad \underline{u}(x) = \liminf_{\substack{\varepsilon\to 0^+\\ y\to x}} u^\varepsilon(y)$$

Notice that $\underline{u} \leq \overline{u}$. We proved that $\overline{u}$ is a viscosity subsolution to the mean curvature equation, $\underline{u}$ is a viscosity supersolution, and $\overline{u}(x) = \underline{u}(x) = 0$ for $x \in \partial\Omega_0$. Therefore, by the comparison principle, we get that $\overline{u} \leq \underline{u}$. Finally, $\underline{u} = \overline{u} = u$. Hence, the limit of the family u^ε exists. That is

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) = u(x).$$

The function u is the unique viscosity solution to the mean curvature equation

Idea of the proof:
Suppose that u is smooth.

$$u(x) \approx \sup_A \inf_B \left\{ \frac{1}{\sigma(A \cap B)} \int_{A \cap B} u(x + v\varepsilon)\, d\sigma(v) \right\} + \varepsilon^2 K,$$

that is

$$0 \approx \sup_A \inf_B \left\{ \frac{1}{\sigma(A \cap B)} \int_{A \cap B} \big(u(x + v\varepsilon) - u(x)\big)\, d\sigma(v) \right\} + \varepsilon^2 K.$$

Using Taylor, and dividing by ε^2 we get

$$0 \approx \sup_A \inf_B \left\{ \frac{1}{\sigma(A \cap B)} \int_{A \cap B} \left(\frac{1}{\varepsilon} \langle \nabla u(x), v \rangle + \frac{1}{2} \langle D^2 u(x) v, v \rangle \right) d\sigma(v) \right\} + K.$$

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Now, assuming that $\nabla u(x) \neq 0$, the leading term is the one that involves $1/\varepsilon$.

- **Paul strategy:** v such that $\langle \frac{\nabla u(x)}{|\nabla u(x)|}, v\rangle$ is as large as possible, and he may choose $A = \{v : \langle \frac{\nabla u(x)}{|\nabla u(x)|}, v\rangle \geq -\theta_\varepsilon\}$ with θ_ε such that $\sigma(A) = \frac{1}{2}\sigma(S^{N-1}) + \delta_\varepsilon$.
- **Carol strategy:** v such that $-\langle \frac{\nabla u(x)}{|\nabla u(x)|}, v\rangle$ is as large as possible, and she may choose $B = \{v : \langle \frac{\nabla u(x)}{|\nabla u(x)|}, v\rangle \leq \theta_\varepsilon\}$.

With these choices of the sets A and B we get $A \cap B = \{v : -\theta_\varepsilon \leq \langle \frac{\nabla u(x)}{|\nabla u(x)|}, v\rangle \leq \theta_\varepsilon\}$ and, in this case, by symmetry, the leading term vanishes, since we have

$$\int_{-\theta_\varepsilon \leq \langle \frac{\nabla u(x)}{|\nabla u(x)|}, v\rangle \leq \theta_\varepsilon} \langle \nabla u(x), v\rangle d\sigma(v) = 0.$$

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Then, with the sets A and B described before, we arrive to

$$0 \approx \frac{1}{\sigma(-\theta_\varepsilon \leq \langle \nabla u(x), v \rangle \leq \theta_\varepsilon)} \int_{-\theta_\varepsilon \leq \langle \nabla u(x), v \rangle \leq \theta_\varepsilon} \frac{1}{2} \langle D^2 u(x) v, v \rangle d\sigma(v) + K.$$

We observe that the sets $A \cap B = \{-\theta_\varepsilon \leq \langle \nabla u(x), v \rangle \leq \theta_\varepsilon\}$ converge to $\{\langle \nabla u(x), v \rangle = 0\}$ as $\varepsilon \to 0$ and hence, we obtain

$$0 = \frac{1}{\mu(\langle \nabla u(x), v \rangle = 0)} \int_{\langle \nabla u(x), v \rangle = 0} \frac{1}{2} \langle D^2 u(x) v, v \rangle d\mu(v) + K.$$

Hence, if we choose $K = C$, we arrive to the mean curvature equation

$$0 = \frac{1}{\mu(\langle \nabla u(x), v \rangle = 0)} \int_{\langle \nabla u(x), v \rangle = 0} \frac{1}{2} \langle D^2 u(x) v, v \rangle d\mu(v) + C.$$

That is

$$\Delta u(x) - \left\langle D^2 u(x) \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right\rangle = -1.$$

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THANK YOU!