Greedy approximation and basis behavior in Hardy spaces of Dirichlet series

Silvia Lassalle

UNIVERSIDAD DE SAN ANDRÉS & CONICET

(Joint work with Daniel Carando & Leandro Milne)

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  - there is a unique sequence $(\mathbf{e}_n^*)_n \subset \mathbb{X}^*$ biorthogonal to $(\mathbf{e}_n)_n$.
  - if $\mathbf{e}_n^*(f) = 0$ for all n, then $f = 0$.

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※ A Schauder basis $\mathcal{B}$ is **unconditional** if there is $\mathbf{C} > 0$ so that for all $f \in \mathbb{X}$, and all $(\varepsilon_n) \subset \mathbb{K}$: $|\varepsilon_n| = 1$,

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Spaces of Dirichlet series

⋇ **The ordinary case**

$$\left\{ D(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \text{with } (a_n) \subset \mathbb{C} \text{ and } s \text{ a complex variable.} \right\}$$

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- **Polynomials** (for a finite set $A \subset \mathbb{N}$):

$$D = \sum_{n \in A} a_n n^{-s} \qquad \text{or} \qquad D = \sum_{n \in A} a_n e^{-\lambda_n s}.$$

For $1 \leq p < \infty$, the **Hardy space of Dirichlet series** is defined as the completion of the space of polynomials with respect to the p-norm

$$\left\|D\right\|_{\mathcal{H}_p} = \left(\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} \left| \sum_{n\in A} a_n n^{-it} \right|^p dt \right)^{\frac{1}{p}}$$

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For $p = \infty$, proceeding similarly, we obtain a subspace of the classical space $\mathcal{H}_\infty$. This space, denoted $\mathcal{A}_\infty$, is the closure of the space of polynomials with respect to the norm

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Hardy spaces of Dirichlet series (the ordinary case)

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The spaces are denoted, respectively, by $\mathcal{H}_p, \mathcal{A}_\infty \subset \mathcal{H}_\infty$ and $\mathcal{H}_p^\lambda, \mathcal{A}_\infty^\lambda \subset \mathcal{H}_\infty^\lambda$.

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  → If λ is $\mathbb{Q}$-linearly independent.
  Carando–Defant–Marceca–Schoolmann (2021) and Schoolmann (2020).

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※ *What can be said about $\mathcal{B}_\lambda = \{e^{-\lambda_n s}\}$ in the context of general Hardy-Dirichlet spaces?*

Fundamental functions

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Super-democracy functions

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For $A \subset \mathbb{N}$, we take the **set of signs**: $\mathcal{E}_A = \emptyset$ if $A = \emptyset$ and

$$\mathcal{E}_A := \{\varepsilon = (\varepsilon_n)_{n \in A} \colon \quad |\varepsilon_n| = 1 \ \forall n \in A\}.$$

Given $\mathcal{B} = \{\mathbf{e}_n\}$ in $\mathbb{X}$, $A \subset \mathbb{N}$ finite, and $\varepsilon \in \mathcal{E}_A$, we consider the **indicator sums**:

$$\mathbb{1}_A := \mathbb{1}_A[\mathcal{B}, \mathbb{X}] = \sum_{n \in A} \mathbf{e}_n, \qquad \text{and} \qquad \mathbb{1}_{\varepsilon,A} := \mathbb{1}_{\varepsilon,A}[\mathcal{B}, \mathbb{X}] = \sum_{n \in A} \varepsilon_n \mathbf{e}_n.$$

Democracy functions

$$\varphi_u(N) := \varphi_u[\mathcal{B}, \mathbb{X}] := \sup\{\|\mathbb{1}_A\| \colon A \subset \mathbb{N}, |A| \leq N\} \qquad \text{(upper)}$$

$$\varphi_l(N) := \varphi_l[\mathcal{B}, \mathbb{X}] := \inf\{\|\mathbb{1}_A\| \colon A \subset \mathbb{N}, |A| \geq N\} \qquad \text{(lower)}$$

Super-democracy functions

$$\varphi_{u,\mathcal{E}}(N) := \varphi_{u,\mathcal{E}}[\mathcal{B}, \mathbb{X}] := \sup\{\|\mathbb{1}_{\varepsilon,A}\| \colon A \subset \mathbb{N}, |A| \leq N, \varepsilon \in \mathcal{E}_A\} \qquad \text{(upper)}$$

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$$\varphi_{l,\mathcal{E}} \ \leq \ \varphi_l \ \leq \ \varphi_u \ \leq \ \varphi_{u,\mathcal{E}}$$

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$$\mathfrak{P} \xrightarrow{\mathfrak{B}} \mathfrak{D}$$

$$\underbrace{f = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha}_{\text{(possitive) Power series}} \xrightarrow{c_\alpha = a_{\mathfrak{p}^\alpha}} \underbrace{D = \sum_n a_n n^{-s}}_{\text{Dirichlet series}}$$

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Connects the **Hardy space** $H_p(\mathbb{T}^\infty)$ with the spaces of **Dirichlet Series** $\mathcal{H}_p$:

$\mathcal{H}_p := \mathfrak{B}(H_p(\mathbb{T}^\infty))$ isometrically (via the norm) $\left\|D\right\|_{\mathcal{H}_p} := \left\|\mathfrak{B}^{-1}(D)\right\|_{H_p(\mathbb{T}^\infty)}$.

$\big(\mathbb{T}^\infty = \prod_{k=1}^\infty \mathbb{T}$ is considered with its normalized Lebesgue measure, which is the countable product measure of the normalized Lebesgue measure on $\mathbb{T}.\big)$

Some examples of estimates of $\left\| \sum_{n \in A} \varepsilon_n n^{-s} \right\|_{\mathcal{H}_p}$, $|\varepsilon_n| = 1$, using:

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- if $p = 2$: $\varphi_{l,\mathcal{E}}^2(N) \approx \varphi_l^2(N) \approx \varphi_u^2(N) \approx \varphi_{u,\mathcal{E}}^2(N) \approx N^{\frac{1}{2}}$.

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Special types of frequencies

$\mathbb{Q}$-linearly independent frequencies.

If λ is $\mathbb{Q}$-linearly independent, then we have, with equivalent norms:

$$\mathcal{H}_p^\lambda = \mathcal{H}_2^\lambda \quad \text{for all} \quad 1 \leq p < \infty \quad \text{and}$$

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Proposition (intermediate behaviors)

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Properties of a basis $\mathcal{B}$ related to greedy algorithms $(C > 0)$

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- Super-democracy $\Longleftrightarrow$ democracy + SUCC ([AABW], 2021).

Democracy and fundamental functions

Given a basis $\mathcal{B}$

- $\mathcal{B}$ is democratic if and only if $\varphi_l(N) \approx \varphi_u(N)$.
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Theorem (λ a frequency)

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or

$$\varphi_{l,\varepsilon}^{\infty,\lambda}(N) \approx N = \varphi_{u,\varepsilon}^{\infty,\lambda}(N).$$

Theorem

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\end{array}$$

The above result says that for λ a frequency containing arbitrarily long arithmetic progressions, $\mathcal{B} = \{e^{-\lambda_n s}\}$ in $\mathcal{H}_p^{\lambda}$ enjoys any of the greedy-type property above if and only if $p = 2$.