

Greedy approximation and basis behavior in Hardy spaces of Dirichlet series

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Given a Banach space \mathbb{X} over a field \mathbb{K} , we say that

- * A sequence $\mathcal{B} = \{\mathbf{e}_n\}_n$ is a **Markushevich basis** for \mathbb{X} if:
 - $\mathbb{X} = \overline{[\mathbf{e}_n : n \in \mathbb{N}]}$.
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 - if $\mathbf{e}_n^*(f) = 0$ for all n , then $f = 0$.

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- * A Schauder basis \mathcal{B} is **unconditional** if there is $\mathbf{C} > 0$ so that for all $f \in \mathbb{X}$, and all $(\varepsilon_n) \subset \mathbb{K}$: $|\varepsilon_n| = 1$,

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* The ordinary case

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✱ **Polynomials** (for a finite set $A \subset \mathbb{N}$):

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Hardy spaces of Dirichlet series (the ordinary case)

For $1 \leq p < \infty$, the **Hardy space of Dirichlet series** is defined as the completion of the space of polynomials with respect to the p -norm

$$\|D\|_{\mathcal{H}_p} = \left(\lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R \left| \sum_{n \in A} a_n n^{-it} \right|^p dt \right)^{\frac{1}{p}}$$

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The spaces are denoted, respectively, by $\mathcal{H}_p, \mathcal{A}_\infty \subset \mathcal{H}_\infty$ and $\mathcal{H}_p^\lambda, \mathcal{A}_\infty^\lambda \subset \mathcal{H}_\infty^\lambda$.

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→ If λ is \mathbb{Q} -linearly independent.

[Carando–Defant–Marceca–Schoolmann \(2021\)](#) and [Schoolmann \(2020\)](#).

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- ✳ What can be said about $\mathcal{B}_\lambda = \{e^{-\lambda n^s}\}$ in the context of general Hardy-Dirichlet spaces?

Fundamental functions

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Super-democracy functions

$$\varphi_{u, \mathcal{E}}(N) := \varphi_{u, \mathcal{E}}[\mathcal{B}, \mathbb{X}] := \sup\{\|\mathbf{1}_{\varepsilon, A}\| : A \subset \mathbb{N}, |A| \leq N, \varepsilon \in \mathcal{E}_A\} \quad (\text{upper})$$

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Fundamental functions

For $A \subset \mathbb{N}$, we take the **set of signs**: $\mathcal{E}_A = \emptyset$ if $A = \emptyset$ and

$$\mathcal{E}_A := \{\varepsilon = (\varepsilon_n)_{n \in A} : |\varepsilon_n| = 1 \ \forall n \in A\}.$$

Given $\mathcal{B} = \{\mathbf{e}_n\}$ in \mathbb{X} , $A \subset \mathbb{N}$ finite, and $\varepsilon \in \mathcal{E}_A$, we consider the **indicator sums**:

$$\mathbf{1}_A := \mathbf{1}_A[\mathcal{B}, \mathbb{X}] = \sum_{n \in A} \mathbf{e}_n, \quad \text{and} \quad \mathbf{1}_{\varepsilon, A} := \mathbf{1}_{\varepsilon, A}[\mathcal{B}, \mathbb{X}] = \sum_{n \in A} \varepsilon_n \mathbf{e}_n.$$

Democracy functions

$$\varphi_u(N) := \varphi_u[\mathcal{B}, \mathbb{X}] := \sup\{\|\mathbf{1}_A\| : A \subset \mathbb{N}, |A| \leq N\} \quad (\text{upper})$$

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$$\varphi_{l, \varepsilon} \leq \varphi_l \leq \varphi_u \leq \varphi_{u, \varepsilon}$$

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$$\alpha = (\alpha_1, \dots, \alpha_k, 0, \dots) \in \mathbb{N}_0^{(\mathbb{N})}$$

The Bohr transform:

$$\mathfrak{P} \xrightarrow{\mathfrak{B}} \mathfrak{D}$$

$$f = \underbrace{\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha}_{\text{(positive) Power series}} \xrightarrow{c_\alpha = a_p^\alpha} \underbrace{D = \sum_n a_n n^{-s}}_{\text{Dirichlet series}}$$

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$$\begin{array}{ccc} \mathfrak{P} & \xrightarrow{\mathfrak{B}} & \mathfrak{D} \\ f = \underbrace{\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha}_{\text{(possible) Power series}} & \xrightarrow{c_\alpha = a_p^\alpha} & D = \underbrace{\sum_n a_n n^{-s}}_{\text{Dirichlet series}} \end{array}$$

Connects the **Hardy space** $H_p(\mathbb{T}^\infty)$ with the spaces of **Dirichlet Series** \mathcal{H}_p :

$$\mathcal{H}_p := \mathfrak{B}(H_p(\mathbb{T}^\infty)) \quad \text{isometrically (via the norm)} \quad \|D\|_{\mathcal{H}_p} := \|\mathfrak{B}^{-1}(D)\|_{H_p(\mathbb{T}^\infty)}.$$

($\mathbb{T}^\infty = \prod_{k=1}^\infty \mathbb{T}$ is considered with its normalized Lebesgue measure, which is the countable product measure of the normalized Lebesgue measure on \mathbb{T} .)

Some examples of estimates of $\left\| \sum_{n \in A} \varepsilon_n n^{-s} \right\|_{\mathcal{H}_p}$, $|\varepsilon_n| = 1$, using:

- $n = p^\alpha = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \iff z \in \mathbb{T}^\infty : z^\alpha = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ and
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Asymptotic estimates of the fundamental functions for $\mathcal{B} = \{n^{-s}\}$ in \mathcal{H}_p

$$\varphi_u^p(N) := \sup_{|A| \leq N} \left\{ \left\| \sum_{n \in A} n^{-s} \right\|_{\mathcal{H}_p} \right\} \quad \varphi_l^p(N) := \inf_{|A| \geq N} \left\{ \left\| \sum_{n \in A} n^{-s} \right\|_{\mathcal{H}_p} \right\},$$

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Remark 1.

- if $p = 2$: $\varphi_{l,\varepsilon}^2(N) \approx \varphi_l^2(N) \approx \varphi_u^2(N) \approx \varphi_{u,\varepsilon}^2(N) \approx N^{\frac{1}{2}}$.

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Asymptotic estimates of the fundamental functions for $\mathcal{B}_\lambda = \{e^{-\lambda_n s}\}$ in \mathcal{H}_p^λ

$$\varphi_u^{p,\lambda}(N) := \sup_{|A| \leq N} \left\{ \left\| \sum_{n \in A} e^{-\lambda_n s} \right\|_{\mathcal{H}_p^\lambda} \right\} \quad \varphi_l^{p,\lambda}(N) := \inf_{|A| \geq N} \left\{ \left\| \sum_{n \in A} e^{-\lambda_n s} \right\|_{\mathcal{H}_p^\lambda} \right\},$$

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Theorem 2.

The fundamental functions of the canonical basis $\mathcal{B} = \{e^{-\lambda n^s}\}$ for \mathcal{H}_p^λ satisfy:

1. Si $p = 1$,
$$\log(N) \lesssim \varphi_{l,\varepsilon}^{1,\lambda}(N) \leq \varphi_l^{1,\lambda}(N) \lesssim N^{\frac{1}{2}}, \quad \varphi_u^{1,\lambda}(N) \approx \varphi_{u,\varepsilon}^{1,\lambda}(N) \approx N^{\frac{1}{2}}.$$

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Special types of frequencies

\mathbb{Q} -linearly independent frequencies.

If λ is \mathbb{Q} -linearly independent, then we have, with equivalent norms:

$$\mathcal{H}_p^\lambda = \mathcal{H}_2^\lambda \quad \text{for all } 1 \leq p < \infty \quad \text{and}$$

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Frequencies containing arbitrarily long arithmetic progressions.

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Proposition (intermediate behaviors)

For every $p > 2$ and every $\frac{1}{2} < t < \frac{1}{p}$ there exists a frequency λ such that

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Properties of a basis \mathcal{B} related to greedy algorithms ($C > 0$)

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- Super-democracy \iff democracy + SUCC ([AABW], 2021).

Democracy and fundamental functions

Given a basis \mathcal{B}

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For $\mathcal{B} = \{n^{-s}\}$ in \mathcal{H}_p ,

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Theorem (λ a frequency)

The basis $\mathcal{B} = \{e^{-\lambda n^s}\}$ of \mathcal{H}_p^λ is SUCC if and only if it is superdemocratic. Moreover, when this holds, we have

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or

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Theorem

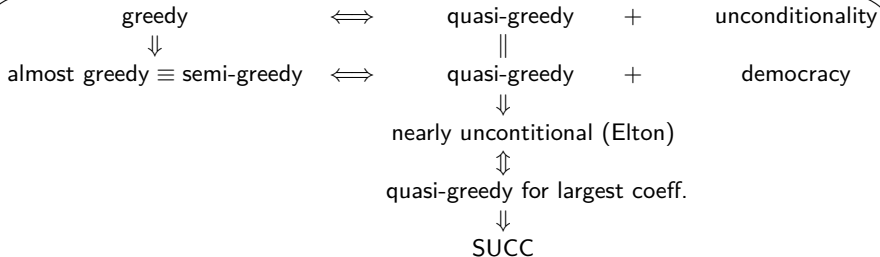
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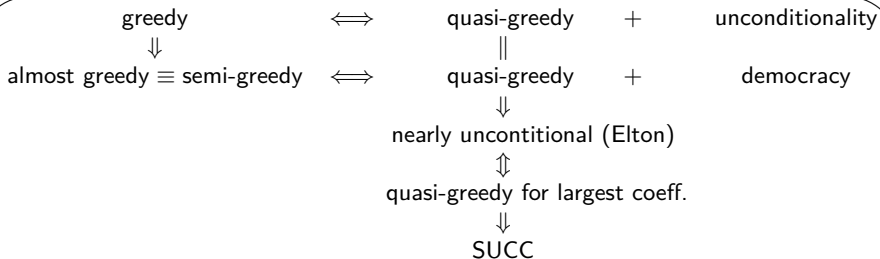
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 Then, $\mathcal{B} = \{e^{-\lambda ns}\}$ in \mathcal{H}_p^λ is SUCC if and only if $p = 2$.



The above result says that for λ a frequency **containing arbitrarily long arithmetic progressions**, $\mathcal{B} = \{e^{-\lambda ns}\}$ in \mathcal{H}_p^λ enjoys any of the greedy-type property above if and only if $p = 2$.