Di Tella Workshop on Analysis and Beyond

Universidad Torcuato Di Tella, Buenos Aires (June 2025)

Prescribed mean curvature surfaces and strictly stable minimal surfaces

Agencia Nacional de Investigación y Desarrollo
Ministerio de Ciencia, Tecnología, Conocimiento e Innovación
Gobierno de Chile

Pedro Gaspar (UC Chile),

joint with J. Marx-Kuo (Rice University)

Supported by ANID - Fondecyt Iniciación

Central question

(M^n, g) Riemannian manifold, $h \in C^\infty(M)$.

How many *closed*, smoothly embedded hypersurfaces $\Sigma^n \subset (M, g)$ with mean curvature $H_\Sigma = h|_\Sigma$ can we find in (M, g)?

Conjecture* (S.T. Yau, '82)

If M is compact, $n = 3$, and $h \equiv 0$ there are **infinitely many** such (minimal) surfaces.

Twin Bubble Conjecture (S.P. Novikov '82, V.I. Arnold '00, X. Zhou '22)

If M is compact, $2 \leq n \leq 7$, and $h \equiv$ constant, there are at least two such hypersurfaces.

Trickier than parametric problem, or problem with boundary conditions...

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Variational methods for minimal surfaces

Known results for $h \not\equiv 0$ & our contributions

Key ideas & ingredients

Variational formulation

Let $\mathcal{C} := \{\Omega \subset M \colon \mathbf{1}_\Omega \in BV(M)\}$ (Caccioppoli sets) and $\mathcal{A}^h \colon \mathcal{C} \to \mathbb{R}$ be:

$$\mathcal{A}^h(\Omega) = \mathrm{Per}(\Omega) - \int_\Omega h \, \mathrm{d}\mathcal{H}^n_g$$

1st variation formula $\Rightarrow$ stationary points of $\mathcal{A}^h$ have $H_{\partial\Omega} = h$ (w.r.t. outward normal) i.e. $\partial\Omega$ is an h-PMC.

Almgren '62 Topology: $\mathcal{C}$ is contractible and quotient $\mathcal{Z} = \mathcal{C}/(\Omega \sim M - \Omega)$, has

$$\pi_1(\mathcal{Z}) \simeq \mathbb{Z}_2, \qquad \pi_i(\mathcal{Z}) = 0, \quad \forall i \geq 2.$$

(that is: $\mathcal{Z}$ looks like an $\mathbb{R}P^\infty$...)

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Min-max theory for $\mathscr{A}^0$ = perimeter

For $h \equiv 0$, the functional $\mathscr{A}^0 : \mathcal{Z} \to \mathbb{R}_+$ is well-defined.

Theorem (Almgren '62, Pitts '81, Schoen-Simon '81 CPAM, Simon-Smith '82)

For closed manifolds (M^n, g) with $n \geq 3$, there exist (at least one) closed, embedded minimal $\Sigma^{n-1} \subset M$ $(\delta \mathscr{A}^0|_\Sigma = 0)$ with optimal regularity.

Strategy (variational methods): Given a homotopically closed family $\mathcal{F}$ of maps $\Phi : X \to \mathcal{Z}$, let

$$\mathbf{L}(\mathcal{F}) = \inf_{\Phi \in \mathcal{F}} \sup_{x \in \mathrm{dom}(\Phi)} \mathrm{Per}(\Phi(x)).$$

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If $\mathbf{L}(\mathcal{F}) > 0$, then $\exists$ a hypersurface Σ with $\delta\mathcal{A}^0|_\Sigma = 0$ and "area$(\Sigma) = \mathbf{L}(\mathcal{F})$".

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If $\mathbf{L}(\mathcal{F}) > 0$, then $\exists$ hypersurfaces $\Sigma_1, \ldots, \Sigma_k$ and positive integers $m_1, \ldots, m_k$ with $\delta\mathcal{A}^0|_{\Sigma_i} = 0$ and

$$m_1 \,\mathrm{area}(\Sigma_1) + \ldots + m_k \,\mathrm{area}(\Sigma_k) = \mathbf{L}(\Pi).$$

Min-max theory for $\mathcal{A}^0$: The volume spectrum

$\mathcal{Z}$ contains a decreasing sequence $\{\mathcal{F}_p\}_{p\in\mathbb{N}}$ of (cohomological) nontrivial families.

Definition (Gromov '86 '03 GAFA, Guth '09 GAFA, Marques-Neves '17 Invent. Math.)

The volume spectrum of (M, g) is the sequence of min-max values:

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- Asymptotics of $\omega_p(M, g)$ & stability (Morse) index bounds;
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Existence of closed PMCs: known results

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- ∞-ly many c-CMCs, or
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Many CMCs from the volume spectrum

A. Dey ’23 J. Differ. Geom.: If $\omega_p(M,g) < \omega_{p+1}(M,g)$ and $0 < c < \frac{\omega_{p+1}-\omega_p}{\mathrm{vol}_g(M)}$, then $\mathcal{A}^c$ has a critical point Ω with $\mathcal{A}^c(\Omega)$ comparable to ω_p.

$\Rightarrow$ any closed (M^n, g) contains $\asymp c^{-\frac{1}{n}}$ closed c-CMC hypersurfaces.

Question

1. What if $\liminf_p \left(\omega_{p+1}(M,g) - \omega_p(M,g)\right) > 0$?
2. What about h-PMCs with nonconstant h?

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Question

1. What if $\liminf_p \left(\omega_{p+1}(M,g) - \omega_p(M,g)\right) > 0$? Impossible for compact M
2. What about h-PMCs with nonconstant h? Replace by $\int_M h \in [0, \omega_{p+1} - \omega_p]$

Strictly stable hypersurfaces & contracting neighborhoods

Let (M^n, g) be compact with $3 \leq n \leq 7$, and $\Sigma = \partial M$ be minimal and strictly stable, that is $\delta\mathcal{A}^0|_\Sigma = 0$ and $\delta^2\mathcal{A}^0|_\Sigma > 0$.

Σ has a contracting neighborhood: $\exists$ a foliation $\Phi\colon \Sigma \times [0, \hat{t}) \to M$ around Σ such that $\Sigma_t = \Phi(\Sigma, t)$ has mean curvature vector pointing to $\Sigma = \Sigma_0$

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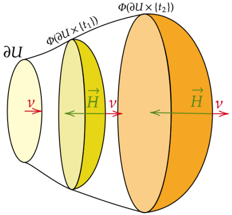

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Main result

Theorem (G.–Marx-Kuo '25 arXiv)

For such (M^n, g) and $\Sigma = \partial M$, let $\Sigma_1 \subset \Sigma$ be the component of least area.

There exist **infinitely many** *closed*, almost embedded h-PMC hypersurfaces, for any $h \in C^\infty(M)$ with $\int_M h > 0$ satisfying:

- (Boundary vanishing) $h|_\Sigma = \partial_\nu h|_\Sigma = 0$
- (Smallness) $\|h\|_{L^1} < \text{area}(\Sigma_1)/2$
- (Genericity) $h|_{\text{int}(M)}$ is a Morse function, and $\{h = 0\} \setminus \Sigma$ is contained in a closed hypersurface whose mean curvature vanishes to at most finite order.

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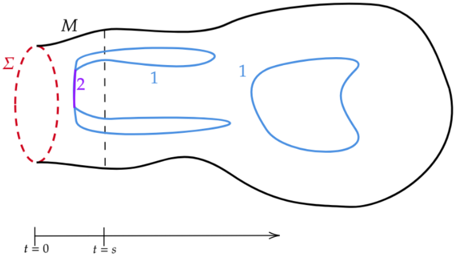

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For any $h \in C^\infty(M)$ satisfying the same assumptions on $\overline{M \setminus \Sigma}$, there are infinitely many distinct, (almost-)embedded, smooth h-PMCs in (M^n, g)

(with the same area growth and *disjoint from* Σ).

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Generically, this can be applied if:

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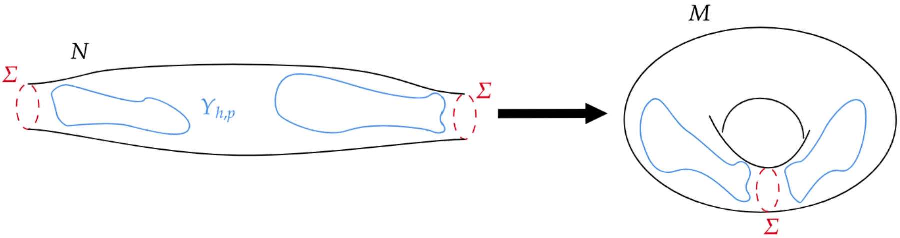

A few key ideas: manifolds with cylindrical ends

Consider the *Lipschitz* manifold with cylindrical ends:

$$\mathrm{Cyl}(M) = M \sqcup (\Sigma \times [0, \infty))\,.$$

Song '23: $\omega_p(\mathrm{Cyl}(M)) \asymp p$; in fact,

$$\omega_{p+1}(\mathrm{Cyl}(M)) - \omega_p(\mathrm{Cyl}(M)) \geq \mathrm{area}(\Sigma_1), \qquad \forall p.$$

Moreover, there exist metrics on $U_\epsilon := M \setminus \Phi(\Sigma \times [0, o(\epsilon)))$ such that $U_\epsilon \to \mathrm{Cyl}(M)$ geometrically.

$\Rightarrow \omega_{i+1}(U_\epsilon) - \omega_i(U_\epsilon) \geq \frac{3}{4}\mathrm{area}(\Sigma_1), \quad i = 1, \ldots, p.$ (for small $\epsilon > 0$)

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A few key ideas: maximum principle and compactness of PMCs

We approximate h in $C^{1,\alpha}$ by compactly supported $h_\epsilon \in C^\infty_c(U_\epsilon)$ satisfying suitable transversality conditions on U_ϵ, so that we can use:

(i) **Sun-Wang-Zhou**'s min-max theory for free-boundary h_ϵ-PMCs in U_ϵ

(ii) the (geometric) **maximum principle** with $(\Sigma \times \{t\})_{t\geq 0}$ as barriers (relies on the compactness of supp h_ϵ);

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¡Muchas gracias!

Extra: the strategy

I. Find relative homotopy classes in $\mathcal{C}(U_\epsilon)$ (**Dey** '20), and use **Sun-Wang-Zhou**'s min-max to find at least p free boundary critical points $\{Y_{\epsilon,i}\}_{i=1}^p$ of $\mathcal{A}^{h_\epsilon}$ in (U_ϵ, g_ϵ), for suitable smooth approximations $h_\epsilon \to h$.

II. Using novel diameter estimates by **Marx-Kuo–Chambers** '24 arXiv, we show $Y_{\epsilon,i}$ have uniformly bounded *extrinsic* diameter.

III. (Smallness) & the MP then push each $Y_{\epsilon,i}$ into core of Cyl(M), away from ∂U_ϵ.

IV. Let $V_{h,i} = \lim_\epsilon Y_{\epsilon,i} \subset M$ (weakly). We use the maximum principle by **Solomon-White** '89 IUMJ, **White** '10 Comm. Anal. Geom. & monotonicity formula to show that no component of $V_{h,i}$ equals any component of Σ.

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