Homogenization of a Local/NonLocal PDE System with Periodic Holes

Luiza Camile Rosa da Silva
IME - Universidade de São Paulo
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Joint work with Marcone Pereira and Julio Rossi

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What is the Homogenization Theory?

The Homogenization Theory of PDE of periodic media consists in analyzing the asymptotic behavior of a Differential Equation when the domain changes. Here we will deal with perforated domains with the following structure:

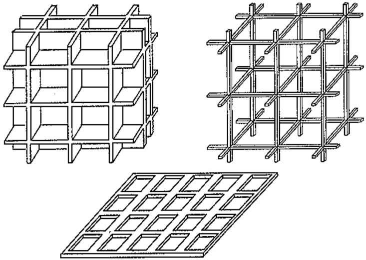

Figure: Perforated Domain

Classical Example with Peridioc Coefficient

For an introduction, let us present a classical example: Let $f \in L^2(\Omega)$ and $a \in L^\infty$ T-periodic with $0 < c_1 \leq a \leq c_2 < \infty$. For each $n \in \mathbb{N}$ consider the following problem with an oscillating coefficient:

$$-\frac{d}{dx}\left(a\left(\frac{x}{n}\right)\frac{du^n}{dx}(x)\right) = f(x).$$

The variational formulation is given by

$$\int_0^1 a\left(\frac{x}{n}\right)\frac{du_n}{dx}\frac{d\varphi}{dx}dx = \int_0^1 \varphi(x)f(x)dx, \qquad \forall\varphi \in H_0^1(\Omega)(P_n)$$

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- u_n converges to some limit u as $n \to \infty$?
- In what sense and in what space do we get this convergence?
- Does u solve a limit equation?

Theorem (Spagnolo(1967))

There exists a unique $u \in H^1(\Omega)$ solution of the homogenized equation

$$-M(a)\frac{d^2u}{dx^2}(x) = f(x),$$

such that $u_n \rightharpoonup_w u \in H_0^1(\Omega)$ and $u_n \to u \in L^2(\Omega)$, with

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- $M(a) = \frac{1}{\frac{1}{T}\int_0^T \frac{ds}{a(s)}}$ *is the homogenized coefficient.*
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Spagnolo, Sergio. (1967)

Cioranescu, D., Saint Jean Paulin, J. (1999).

Luc Tartar (2009)

Uniform boundedness for u_n is obtained from *Poincaré's inequality* taking $\varphi = u^\epsilon$ in (P^n):

$$C\|u_n\|^2_{L^2(0,1)} \leq \alpha \left\| \frac{du_n}{dx} \right\|^2_{L^2(0,1)} \leq \left\| \sqrt{a_n} \frac{du_n}{dx} \right\|^2_{L^2(0,1)} \leq \|f\|_{L^2(0,1)} \|u_n\|_{L^2(0,1)}.$$

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The crucial step in the proof is: Passing to the limit in the variational formulation of the problem,

$$\int_0^1 a\left(\frac{x}{n}\right)\frac{du_n}{dx}(x)\frac{d\varphi}{dx}(x)\,dx = \int_0^1 \varphi(x)f(x)\,dx \quad \forall \varphi \in H_0^1(0,1). \qquad (P^n)$$

to obtain

$$\int_0^1 M(a)\frac{du}{dx}(x)\frac{d\varphi}{dx}(x)\,dx = \int_0^1 \varphi(x)f(x)\,dx \quad \forall \varphi \in H_0^1(0,1).$$

Remark

The difficulty here is passing to the limit in the product of weak convergences in P^n!!!

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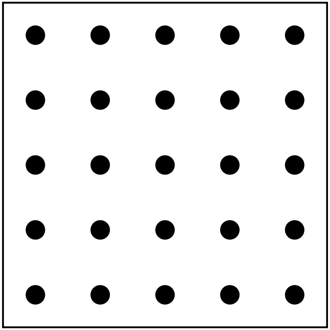
Setting of the domain

To formulate our homogenization p´problem, we fix $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a bounded domain and consider a **disjoint partition**

$$\Omega = A_n \cup B_n$$

which will be distributed in the following way:



System of Local-Nonlocal Equations

The kernels. Let $J, G : \mathbb{R}^N \to \mathbb{R}$ be smooth probability kernels, compactly supported and radially symmetric. We will deal with the model:

$$\begin{cases} f(x) = \Delta u_n(x) + \displaystyle\int_{B_n} J(x-y)(v_n(y) - u_n(x))dy, & x \in A_n \\ \dfrac{\partial u_n}{\partial n}(x) = 0, & x \in \partial A_n. \end{cases}$$

$$g(x) = \int_{B_n} G(x-y)(v_n(y)-v_n(x))dy + \int_{A_n} J(x-y)(u_n(y)-v_n(x))dy, \; x \in B_n.$$

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$$g(x) = \int_{B_n} G(x-y)(v_n(y)-v_n(x))dy + \int_{A_n} J(x-y)(u_n(y)-v_n(x))dy, \; x \in B_n.$$

For similar systems we refer to

Capanna, M., Nakasato, J. C., Pereira, M. C., Rossi, J. D.

.Cassol dos Santos, B., Oliva, S. M., Rossi, J. D.

Existence and uniqueness of solutions

Assume that $\int_A f(x)dx + \int_B g(x)dx = 0$. This model is associated with an energy

$$E_n(u_n, v_n) = \frac{1}{2}\int_{A_n} |\nabla u_n|^2(x)dx + \frac{1}{4}\int_{B_n}\int_{B_n} G(x-y)(v_n(y) - v_n(x))^2 dxdy$$
$$+ \frac{1}{2}\int_{A_n}\int_{B_n} J(x-y)(v_n(y) - u_n(x))^2 dydx$$
$$+ \int_{A_n} f(x)u_n(x)dx + \int_{B_n} g(x)v_n(x)dx,$$

which, for each n, has a unique minimizer in the space

$$W = \left\{(u,v) \in H^1(A) \times L^2(B); \int_A u(x)dx + \int_B v = 0\right\}.$$

For the homogenization of the system, we observe that we have weak convergence of the characteristic functions of A_n, B_n as $n \to \infty$. It holds that

$$\chi_{A_n}(x) \rightharpoonup X, \quad \text{weakly-* in } L^\infty(\Omega),$$

and

$$\chi_{B_n}(x) \rightharpoonup 1 - X, \quad \text{weakly-* in } L^\infty(\Omega),$$

where X satisfies the bounds:

$$1 > 1 - c_1 \geq X \geq c_0 > 0 \tag{1}$$

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Theorem (Local in Balls)

There exists a unique pair $(u_A, v_B) \in L^2(\Omega) \times L^2(\Omega)$ *such that*

$$\chi_{A_n} u_n \rightharpoonup u_A \in L^2(A) \quad \text{and} \quad \chi_{B_n} u_n \rightharpoonup u_B \in L^2(B).$$

The pair (u_A, v_B) *is characterized as being the unique solution of the following system:*

$$f(x)X = \int_\Omega J(x-y)[v_B(y)X(x) - (1 - X(y))u_A(x)]dy.$$

$$g(x)(1-X) = \int_\Omega G(x-y)((1-X)v_B(y) - (1-X)v_B(x))dy$$

$$+ \int_\Omega J(x-y)((1-X)u_A(y) - Xv_B(x))dy,$$

Key idea

The proof relies on taking special test functions $\{\varphi_n : \Omega \to \mathbb{R}\}$ in the weak formulation of the system.
Let $\varphi \in C(\overline{\Omega})$ and define

$$\varphi_n(x) = \begin{cases} \varphi(x), & x \in B_n \\ \fint_{B_{rj}(x_j)} \varphi(y)dy, & x \in B_{r_n}(x_j). \end{cases}$$

Notice that we get $\varphi_n \in H^1(A_n)$ with vanishing gradients inside $A_n = \cup_{j=1}^{k} B_{r_n}(x_j)$ and such that

$$\varphi_n \to \varphi \qquad \text{as } n \to \infty,$$

uniformly in $C(\Omega)$.

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Theorem (Non-Local in Balls)

There exists a unique $(u_A, v_B) \in L^2(\Omega) \times L^2(\Omega)$ *such that*

$$\chi_{A_n} u_n \rightharpoonup X u_A \in L^2(A) \quad \text{and} \quad \chi_{B_n} v_n \rightharpoonup u_B \in L^2(B)$$

The pair (u_A, v_B) *is characterized as being the unique solution of the following system:*

$$f(x)X = q_{ij}\Delta u_A(x) + \int_\Omega J(x-y)X[(1-X)u_A(x) - v_B(x)X]dy,$$

$$g(x)(1-X) = \int_\Omega G(x-y)((1-X)v_B(x) - (1-X)v_B(y))dy$$
$$- \int_\Omega J(x-y)X(u_A(y)(1-X) - Xv_B(x))dy,$$

where $q_{i,j}$ *are the classical homogenization coefficients.*

Key idea

The proof relies on considering correctors of the form

$$w_n(x) = \begin{cases} u_A(x) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{N} U_i(nx)\dfrac{\partial u_A}{\partial x_i}(x) & x \in A_n, \\ \dfrac{\chi_{B_n} u_B(x)}{1-X} & x \in B_n. \end{cases} .$$

Here U_i are solutions to

$$\begin{cases} -\Delta U_i = 0 & \text{in } Y^* \\ \frac{\partial U_i}{\partial \eta} = \eta_i & \text{on } \partial Y^* \\ U_i & Y^* - \text{periodic.} \end{cases}$$

in the unit cell Y^*.

Thanks!!! :)