Hypercontractivity on classical and quantum Boolean cubes: a tensor norm approach

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Contractivity

Definition

A linear map $T : X \to Y$ between normed spaces is said to be **contractive** if $\|T\| \leq 1$, that is,

$$\text{for every } x \in X \quad \|Tx\| \leq \|x\| \, .$$

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Example

If $1 \leq p \leq q \leq \infty$ and μ is a probability measure, the identity $L_q(\mu) \to L_p(\mu)$ is contractive: by Hölder's inequality

$$\|f\|_{L_p} = \|f \cdot 1\|_{L_p} \leq \|f\|_{L_q} \|1\|_{L_r(\mu)} = \|f\|_{L_q}$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

Hypercontractivity

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If $1 \leq p \leq q \leq \infty$ and μ is a probability measure, the identity $L_q(\mu) \to L_p(\mu)$: by Hölder's inequality

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Definition

If $1 \leq p \leq q \leq \infty$ and μ is a probability measure, a contractive operator "going the other way"

$$T : L_p(\mu) \to L_q(\mu)$$

is said to be **hypercontractive.**

The Boolean cube

Definition

Given $n \in \mathbb{N}$, the **Boolean cube of dimension** n is $\{-1,1\}^n$, that is, the set of vectors $x = (x_1, \ldots, x_n)$ such that $x_j = \pm 1$ for all $j = 1, \ldots, n$.

Definition

By $L_p(\{-1,1\}^n)$ we mean the L_p space associated to the uniform measure on $\{-1,1\}^n$.

Note

For a fixed n these are all equal as vector spaces, i.e. the set of complex-valued functions $f : \{-1,1\}^n \to \mathbb{C}$, that is, $\mathbb{C}^{\{-1,1\}^n} \equiv \mathbb{C}^{2^n}$.

The noise or bit-flip operator

For $\delta \in [-1,1]$, given an n-bit string $x \in \{-1,1\}^n$ define the distribution $y \sim_\delta x$ as follows: independently, each bit of y is equal to the corresponding bit of x with probability $1/2 + \delta/2$, and the bit is flipped with probability $1/2 - \delta/2$.

Definition

The **noise operator** $T_\delta^n : \mathbb{C}^{\{-1,1\}^n} \to \mathbb{C}^{\{-1,1\}^n}$ is given by

$$(T_\delta^n f)(x) = \mathbb{E}_{y \sim_\delta x} f(y)$$

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Hypercontractivity (Carlen-Lieb 1993)

If $1 < p \leq q < \infty$ and $0 \leq \delta \leq \sqrt{\frac{p-1}{q-1}}$, then

$$\left\| T_\delta^n : L_p(\{-1, 1\}^n) \to L_q(\{-1, 1\}^n) \right\| \leq 1$$

Understanding T_δ^1

Definition

The **noise operator** $T_\delta^1 : \mathbb{C}^{\{-1,1\}} \to \mathbb{C}^{\{-1,1\}}$ is given by

$$(T_\delta^1 f)(x) = \mathbb{E}_{y \sim_\delta x} f(y)$$

- Since T_δ^1 is linear, it suffices to understand what it does on a basis.
- If $f_0 = (1,1)$, clearly $T_\delta^1 f_0 = f_0$.
- If $f_1 = (-1,1)$, that is, $f_1(-1) = -1$ and $f_1(1) = 1$:
  - When $x = 1$,

$$\mathbb{P}(y = 1) = 1/2 + \delta/2 \text{ and } \mathbb{P}(y = -1) = 1/2 - \delta/2, \text{ so}$$
$$(T_\delta^1 f_1)(1) = (1/2 + \delta/2) - (1/2 - \delta/2) = \delta$$

  - When $x = -1$,

$$\mathbb{P}(y = -1) = 1/2 + \delta/2 \text{ and } \mathbb{P}(y = 1) = 1/2 - \delta/2 \text{ so}$$
$$(T_\delta^1 f_1)(-1) = -(1/2 + \delta/2) + (1/2 - \delta/2) = -\delta$$

Thus, $T_\delta^1 f_1 = \delta f_1$.

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$$(T_\delta^1 f)(x) = \mathbb{E}_{y \sim_\delta x} f(y)$$

Recall we calculated

$$\begin{aligned} f_0 &= (1,1), & f_1 &= (-1,1) \\ T_\delta^1 f_0 &= f_0, & T_\delta^1 f_1 &= \delta f_1 \end{aligned}$$

For an arbitrary $f \in \mathbb{C}^{\{-1,1\}}$, writing $f = c_0 f_0 + c_1 f_1$ yields

$$T_\delta^1 f = c_0 f_0 + c_1 \delta f_1 = \delta f + (1-\delta) c_0 f_0$$

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$$\begin{aligned} T_\delta^1 f &= c_0 f_0 + c_1 \delta f_1 = \delta f + (1-\delta) c_0 f_0 \\ &= \delta f + (1-\delta)\frac{(f(1)+f(-1))}{2}(1,1) \end{aligned}$$

Hypercontractivity, $n = 1$

2-point inequality (Bonami 1970, Gross 1975, Beckner 1975)

If $1 < p \leq q < \infty$ and $0 \leq \delta \leq \sqrt{\frac{p-1}{q-1}}$, then

$$\left\| T_\delta^1 : L_p(\{-1,1\}) \to L_q(\{-1,1\}) \right\| \leq 1$$

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Usual strategy of proof for the general hypercontractivity

- Start with $n = 1$.
- Use an inductive argument.

The “tensor product trick”

Crucial observation

Because each bit is flipped independently,

$$T_\delta^n : \mathbb{C}^{\{-1,1\}^n} \to \mathbb{C}^{\{-1,1\}^n}$$

can be identified with

$$\left(T_\delta^1\right)^{\otimes n} : \left(\mathbb{C}^{\{-1,1\}}\right)^{\otimes n} \to \left(\mathbb{C}^{\{-1,1\}}\right)^{\otimes n}$$

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A dream

Given $1 \leq p \leq q \leq \infty$ and operators

$$T_1 : L_p(\mu_1) \to L_q(\nu_1), \qquad T_2 : L_p(\mu_2) \to L_q(\nu_2),$$

do we have

$$\left\| T_1 \otimes T_2 : L_p(\mu_1 \times \mu_2) \to L_q(\nu_1 \times \nu_2) \right\| \leq \|T_1\| \, \|T_2\| ?$$

Tensor products of normed spaces

Given two normed spaces X and Y, there are many possible norms on $X \otimes Y$ which are compatible with the original norms.

Definition

A **tensor norm** α is an assignment, for every pair of normed spaces X and Y, of a norm α on $X \otimes Y$ such that:

(a) For any $x \in X$, $y \in Y$ we have $\alpha(x \otimes y) = \|x\| \, \|y\|$.

(b) For every $x' \in X'$, $y' \in Y'$ we have $\|x' \otimes y' : X \otimes_\alpha Y \to \mathbb{K}\| = \|x'\| \, \|y'\|$.

(c) It satisfies the *metric mapping property*: for any operators

$$T_1 : X_1 \to Y_1, \qquad T_2 : X_2 \to Y_2$$

we have

$$\left\| T_1 \otimes T_2 : X_1 \otimes_\alpha X_2 \to Y_1 \otimes_\alpha Y_2 \right\| \le \|T_1\| \, \|T_2\| \, .$$

The smallest tensor norm ε

For $u \in X \otimes Y$ we define

$$\varepsilon(u) = \sup \left\{ |(x' \otimes y')u| \ : \ x' \in X', y' \in Y', \left\|x'\right\|, \left\|y'\right\| \leq 1 \right\}$$

Which means we have isometric embeddings

$$X \otimes_\varepsilon Y \hookrightarrow \mathcal{L}(X', Y), \quad X \otimes_\varepsilon Y \hookrightarrow \mathcal{L}(Y', X)$$

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Example: $\ell_\infty^n \otimes_\varepsilon Y \equiv \ell_\infty^n(Y)$

Write $u = \sum_{j=1}^n e_j \otimes y_j$, then

$$\varepsilon(u) = \|T_u : \ell_1^n \to Y\|$$

where $T_u((a_j)_{j=1}^n) = \sum_{j=1}^n a_j y_j$, so $\|T_u\| = \max_{1 \le j \le n} \|y_j\|$.

The largest tensor norm π

For $u \in X \otimes Y$ we define

$$\pi(u) = \inf \left\{ \sum_{j=1}^{m} \|x_j\| \, \|y_j\| \ : \ x_j \in X, y_j \in Y, u = \sum_{j=1}^{m} x_j \otimes y_j \right\}$$

which means that the unit ball $B_{X \otimes_\pi Y}$ is the absolutely convex hull of $B_X \otimes B_Y$.

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Example: $\ell_1^n \otimes_\pi Y \equiv \ell_1^n(Y)$

The unit ball of ℓ_1^n is the absolutely convex hull of $\{e_j\}_{j=1}^n$, hence the unit ball of $\ell_1^n \otimes_\pi Y$ is the absolutely convex hull of $\{e_j \otimes y \; : \; 1 \leq j \leq n, \|y\| \leq 1\}$, and that is precisely the unit ball of $\ell_1^n(Y)$.

A first multiplicativity result

Proposition

Given operators

$$T_1 : \ell_1^{n_1} \to \ell_\infty^{m_1}, \qquad T_2 : \ell_1^{n_2} \to \ell_\infty^{m_2},$$

we have

$$\left\| T_1 \otimes T_2 : \ell_1^{n_1 n_2} \to \ell_\infty^{m_1 m_2} \right\| \leq \|T_1\| \, \|T_2\|$$

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Proof

$$\begin{tikzcd}
\ell_1^{n_1 n_2} \arrow[rr, "T_1 \otimes T_2"] \arrow[d] & & \ell_\infty^{m_1 m_2} \\
\ell_1^{n_1} \otimes_\pi \ell_1^{n_2} \arrow[r, "T_1 \otimes T_2"'] & \ell_\infty^{m_1} \otimes_\pi \ell_\infty^{m_2} \arrow[r] & \ell_\infty^{m_1} \otimes_\varepsilon \ell_\infty^{m_2} \arrow[u]
\end{tikzcd}$$

The Chevet-Saphar tensor norms

Definition (Chevet 1969, Saphar 1972)

For $1 < p < \infty$, there is a tensor norm d_p such that the map

$$e \otimes x \otimes f \otimes y \mapsto \langle e, f \rangle x \otimes y$$

gives a quotient

$$(\ell_{p'} \widehat{\otimes}_\varepsilon X) \widehat{\otimes}_\pi \ell_p(Y) \twoheadrightarrow X \widehat{\otimes}_{d_p} Y$$

Properties

- (a) $L_p(\mu_1) \widehat{\otimes}_{d_p} L_p(\mu_2) \equiv L_p(\mu_1 \times \mu_2)$ via the formal identity.
- (b) For any normed spaces X and Y and $1 < p \leq q < \infty$,

$$\left\| X \otimes_{d_p} Y \to X \otimes_{d_q} Y \right\| \leq 1$$

The general multiplicativity result

Proposition

Given $1 < p \leq q < \infty$ *and operators*

$$T_1 : L_p(\mu_1) \to L_q(\nu_1), \qquad T_2 : L_p(\mu_2) \to L_q(\nu_2),$$

we have

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\end{tikzcd}$$

From the classical to the quantum world

The change (from a naïve mathematical point of view)

Replace vectors by matrices

$$\mathbf{x} = (x_1, \dots, x_n) \quad \mapsto \quad \mathrm{diag}(\mathbf{x}) := \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}$$

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Noncommutative ℓ_p spaces

Note that

$$\|\mathbf{x}\|_{\ell_p^n} = \|\operatorname{diag}(\mathbf{x})\|_{S_p^n}$$

where S_p^n denotes the Schatten p-norm.

A Quantum Information dictionary

Classical	**Quantum**
Finite set X	Fin. dim. Hilbert space H
$\lvert X\rvert$	$\dim H$
$X_1 \times X_2$	$H_1 \otimes H_2$
$f : X \to \mathbb{C}$	$A \in \mathcal{L}(H)$
$f : X \to \mathbb{R}$	$A \in \mathcal{L}(H), A = A^*$
$f \geq 0$	$A \geq 0$
$\sum_{x\in X} f(x)$	$\mathrm{tr}(A)$
$\lVert f\rVert_{\ell_1} = \sum_{x\in X} \lvert f(x)\rvert$	$\lVert A\rVert_{S_1} = \mathrm{tr}(\lvert A\rvert)$
$\lVert f\rVert_{\ell_p} = \Big(\sum_{x\in X} \lvert f(x)\rvert^p \Big)^{1/p}$	$\lVert A\rVert_{S_p} = \mathrm{tr}(\lvert A\rvert^p)^{1/p}$

The quantum Boolean cube

Classical	**Quantum**
$\{-1,1\}$	$\mathbb{C}^2$
$\{-1,1\}^n$	$(\mathbb{C}^2)^{\otimes n} \equiv \mathbb{C}^{2^n}$
$\mathbb{C}^{\{-1,1\}^n}$	$\mathcal{L}((\mathbb{C}^2)^{\otimes n}) \equiv \mathcal{L}(\mathbb{C}^2)^{\otimes n} \equiv M_2^{\otimes n} \equiv M_{2^n}$
$\sum_{x\in\{-1,1\}^n} \lvert f(x)\rvert$	$\operatorname{tr}(\lvert A\rvert)$
$\left(\frac{1}{2^n}\sum_{x\in\{-1,1\}^n} \lvert f(x)\rvert^p\right)^{1/p}$	$\left(\frac{1}{2^n}\operatorname{tr}(\lvert A\rvert^p)\right)^{1/p}$

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Denote $\tau_n = \frac{1}{2^n}\,\mathrm{tr}$ (normalized trace on M_{2^n}), and $L_p(\tau_n)$ the corresponding "noncommutative L_p space".

Quantum hypercontractivity?

Proposition (“Noncommutative Hölder”)

If $1 \leq p \leq q \leq \infty$, $\|L_q(\tau_n) \to L_p(\tau_n)\| \leq 1$.

Recall

$T^1_\delta : \mathbb{C}^2 \to \mathbb{C}^2$ is given by

$$T^1_\delta f = \delta f + (1-\delta)\frac{(f(1)+f(-1))}{2}(1,1)$$

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Definition (Montanaro-Osborne 2010)

Define $T^1_{Q,\delta} : M_2 \to M_2$ by

$$T^1_{Q,\delta}(A) = \delta A + (1-\delta)\frac{\mathrm{tr}(A)}{2} I_2$$

where $I_2 \in M_2$ is the identity matrix.

Quantum noise operator

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Proof

- Case $n = 1$ reduces to T^1_δ.
- **The tensor product trick does not work**, they gave a different inductive argument.

Failure of multiplicativity

Fact

Given $1 \le p \le q \le \infty$ and operators

$$T_1 : S_p^{n_1} \to S_q^{m_1}, \qquad T_2 : S_p^{n_2} \to S_q^{m_2},$$

in general we do not have

$$\left\| T_1 \otimes T_2 : S_p^{n_1 n_2} \to S_q^{m_1 m_2} \right\| \le \|T_1\| \, \|T_2\|$$

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Example

$$Id_n : M_n \to M_n, \qquad T : M_n \to M_n \text{ transposition}$$

then $\|Id_n\| = \|T\| = 1$, but

$$\left\| Id_n \otimes T : M_{n^2} \to M_{n^2} \right\| = n$$

A different "operator norm"

Definition

Given $T : M_n \to M_m$, its **completely bounded norm** is

$$\|T\|_{\mathrm{cb}} = \sup_{k\in\mathbb{N}} \|Id_k \otimes T : M_{kn} \to M_{km}\|$$

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Proposition

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$$T_1 : M_{n_1} \to M_{m_1}, \qquad T_2 : M_{n_2} \to M_{m_2},$$

we have

$$\left\|T_1 \otimes T_2 : M_{n_1 n_2} \to M_{m_1 m_2}\right\|_{\mathrm{cb}} \leq \|T_1\|_{\mathrm{cb}} \|T_2\|_{\mathrm{cb}} .$$

Operator spaces

Definition

Let H be a Hilbert space. A **(concrete) operator space** is a linear subspace $X \subseteq \mathcal{L}(H)$, together with the norms on $M_n(X)$ defined via the identification

$$M_n(\mathcal{L}(H)) \equiv \mathcal{L}(H \oplus \cdots \oplus H)$$

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$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}$$

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Fact

By results of Ruan and Pisier, there is a canonical way to do this for the spaces S_p^n, $L_p(\tau_n)$.

General completely bounded maps

Definition

Let $T : X \to Y$ be a linear map. Its **n-th amplification** is the map $T_n : M_n(X) \to M_n(Y)$ given by applying T entrywise, i.e.

$$T_n \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} T(x_{11}) & T(x_{12}) & \cdots & T(x_{1n}) \\ T(x_{21}) & T(x_{22}) & \cdots & T(x_{2n}) \\ \vdots & \vdots & & \vdots \\ T(x_{n1}) & T(x_{n2}) & \cdots & T(x_{nn}) \end{bmatrix}$$

With the identification $M_n(X) = M_n \otimes X$, we have $T_n = Id_n \otimes T$.

Definition

The **completely bounded norm** of a linear map $T : X \to Y$ is

$$\|T\|_{\mathrm{cb}} = \sup_{k \in \mathbb{N}} \left\| T_k : M_k(X) \to M_k(Y) \right\|$$

Operator space tensor norms

Definition

An **operator space tensor norm** α is an assignment, for every pair of operator spaces X and Y, of an operator space structure α on $X \otimes Y$ such that:

(a) (Technical compatibility conditions)

(b) It satisfies the *complete metric mapping property*: for any operators

$$T_1 : X_1 \to Y_1, \qquad T_2 : X_2 \to Y_2$$

we have

$$\left\| T_1 \otimes T_2 : X_1 \otimes_\alpha X_2 \to Y_1 \otimes_\alpha Y_2 \right\|_{\mathrm{cb}} \leq \|T_1\|_{\mathrm{cb}} \, \|T_2\|_{\mathrm{cb}} \, .$$

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(a) There exist a smallest one (min) and largest one (proj).

(b) $M_n \otimes_{\min} X \equiv M_n(X), \qquad S_1^n \otimes_{\mathrm{proj}} X \equiv S_1^n(X).$

The Chevet-Saphar operator space tensor norms

Definition (CD 2016)

For $1 < p < \infty$, there is an operator space tensor norm d_p^o such that for any operator spaces X and Y the map

$$e \otimes x \otimes f \otimes y \mapsto \langle e, f \rangle x \otimes y$$

gives a complete quotient

$$(S_{p'} \widehat{\otimes}_{\min} X) \widehat{\otimes}_{\text{proj}} S_p[Y] \twoheadrightarrow X \widehat{\otimes}_{d_p^o} Y$$

Properties

- (a) $S_p^{n_1} \widehat{\otimes}_{d_p^o} S_p^{n_2} \equiv S_p^{n_1 n_2}$ completely isometrically.
- (b) For any operator spaces X and Y and $1 < p \leq q < \infty$,

$$\left\| X \otimes_{d_p^o} Y \to X \otimes_{d_q^o} Y \right\| \leq 1$$

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We don't know if the same is true for cb-norm.

A weak multiplicativity result

Proposition (CD-Dimant-Galicer 2025)

Given $1 < p \leq q < \infty$ *and operators*

$$T_1 : S_p^{n_1} \to S_q^{m_1}, \qquad T_2 : S_p^{n_2} \to S_q^{m_2},$$

we have

$$\left\| T_1 \otimes T_2 : S_p^{n_1 n_2} \to S_q^{m_1 m_2} \right\| \leq \|T_1\|_{\mathrm{cb}} \|T_2\|_{\mathrm{cb}}$$

Proof

$$\begin{tikzcd}
S_p^{n_1 n_2} \arrow[rr, "T_1 \otimes T_2"] \arrow[d] & & \to S_q^{m_1 m_2} \\
S_p^{n_1} \otimes_{d_p^o} S_p^{n_1} \arrow[r, "T_1 \otimes T_2"'] & S_q^{m_1} \otimes_{d_p^o} S_q^{m_2} \arrow[r] & S_q^{m_1} \otimes_{d_q^o} S_q^{m_2} \arrow[u]
\end{tikzcd}$$

(Complete) Positivity to the rescue

Definition

A linear map $T : M_n \to M_m$ is **positive** if it maps positive semidefinite matrices to positive semidefinite matrices, and it is **completely positive** if $Id_{M_k} \otimes T : M_{kn} \to M_{km}$ is positive for all $k \in \mathbb{N}$.

Theorem (Devetak-Junge-King-Ruskai 2006)

If $T : M_n \to M_m$ *is completely positive and* $1 \leq p \leq q \leq \infty$, *then*

$$\left\| T : S_p^n \to S_q^m \right\|_{\mathrm{cb}} = \left\| T : S_p^n \to S_q^m \right\| .$$

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Luckily

For $\delta \in [0, 1]$, $T^1_{Q,\delta} : M_2 \to M_2$ is completely positive.

Putting everything together

Completely bounded quantum hypercontractivity

If $1 < p \leq q < \infty$ and $\delta = \sqrt{\frac{p-1}{q-1}}$, then

$$\left\| T^n_{Q,\delta} : L_p(\tau_n) \to L_q(\tau_n) \right\|_{\mathrm{cb}} \leq 1$$

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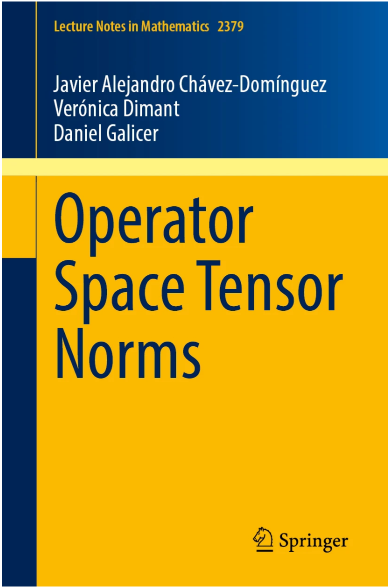
Note

This is stronger than Montanaro-Osborne, and the proof does use the "tensor product trick".

Shameless advertisement



Lecture Notes in Mathematics 2379

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Operator Space Tensor Norms

Springer

THANKS!