Quantitative stability for Yamabe minimizers on manifolds with boundary

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Di Tella Workshop on Analysis and Beyond

The author was partially supported by Fondecyt/ANID

Motivation

- Goal: Study of *quantitative stability* of Geometric and functional inequalities;

General question

*If a function nearly achieves equality (in one inequality), is it possible to show-in a quantitative manner- that the function is close, in a suitable sense, to a corresponding **minimizer**?*

Example: Sobolev inequality

Sobolev inequality, 1938

If $n \geq 3$, we have that

$$\|\nabla \varphi\|_2^2 \geq S_n \|\varphi\|_{2^*}^2 \quad \forall \varphi \in \dot{H}^1(\mathbb{R}^n)$$

where the constant S_n is the best constant.

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- S_n can be *variationally* characterized as

$$S_n = \inf_{u\in\dot{H}^1(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}}$$

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- Computed explicetly in Brezis-Lieb'85, that obtained

$$S_n = \pi n(n-2)[\Gamma(n/2)/\Gamma(n)]^{2/n}$$

- The **extremal functions** are known as bubbles or Aubin-Talenti functions and given by

$$\varphi(x) = \left(\frac{1}{1 + |x|^2} \right)^{\frac{n-2}{2}} \qquad x \in \mathbb{R}^n.$$

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Can we study this inequality in a quantitative way?

- **Translating:** Can the difference

$$\frac{\|\nabla \varphi\|_2^2}{\|\varphi\|_{2^*}^2} - S_n \quad \text{(Sobolev inequality)}$$

control the distance of φ to the set of extremal functions/minimizers?

Bianchi-Egnell inequality

Theorem (Bianchi, Egnell'91 JFA):

There is a constant $C > 0$, that depends only on the dimension, such that:

$$\|\nabla\varphi\|_2^2 - S_n\|\varphi\|_{2^*}^2 \geq C d(\varphi, \mathcal{M})^2 \quad \forall \varphi \in \dot{H}^1(\mathbb{R}^n)$$

where

$$d(\varphi, \mathcal{M}) = \inf_{u \in \mathcal{M}} \|\nabla(\varphi - u)\|_2$$

- $\mathcal{M}$ = set of minimizers or extremal functions;

Historically

This question can be made sense for several useful inequalities:

- **Isoperimetric inequality**: Fusco,Maggi, and Pratelli'08, Annals, Figalli, Maggi, and Pratelli'10, Inventiones.

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- **Gagliardo-Niremberg-Sobolev**: Bonforte, Dolbeault, Nazaret, Simonov'20.

Returning to Bianchi-Egnell: Conformal equivalence

- Stereographic projection: Conformal equivalence between the $\mathbb{S}^n$ and $\mathbb{R}^n$.

There is a constant $C > 0$ such that

$$Q_{(\mathbb{S}^n,g_0)}(\varphi) - Y\left(\mathbb{S}^n\right) \geq C\left(\frac{\inf\left\{\|\varphi - u\|_{H^1(\mathbb{S}^n)} : u \in \mathcal{M}_{(\mathbb{S}^n,g_0)}\right\}}{\|\varphi\|_{H^1(\mathbb{S}^n)}}\right)^2$$

for every function $\varphi \in H^1\left(\mathbb{S}^n\right)$.

- Who is Q and $Y(S^n)$?

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for every function $\varphi \in H^1\left(\mathbb{S}^n\right)$.

- Who is Q and $Y(S^n)$? Connected with the Yamabe energy of the sphere!

From conformal geometry: The Yamabe problem

Let (M, g) be a closed Riemannian manifold and define the Yamabe energy

$$Q(u) := \frac{4\frac{(n-1)}{(n-2)} \int_M |\nabla_g u|^2 \, dv + \int_M R_g u^2 dv}{\left(\int_M |u|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}}}$$

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Critical points of the energy are solutions of the equation

$$\Delta_g u - \frac{n-2}{4(n-1)} R_g u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0$$

- **Geometric interpretation**: Positive solutions $u > 0$, imply the existence of a new metric for M with constant scalar curvature!

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Theorem (Engelstein, Neumayer y Spolaor'22 Transactions of AMS)

Let (M^n, g) be a closed manifold, $n \geq 3$, not conformally equivalent to $\mathbb{S}^n$. There is a constant $C > 0$ y $\gamma \geq 0$ such that

$$Q(\varphi) - Y(M, [g]) \geq C d(\varphi, \mathcal{M})^{2+\gamma} \quad \forall \varphi \in H^1(M, \mathbb{R}_+)$$

- **Conclusion**: Stability discussion to the Yamabe functional!
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Moreover, $\gamma = 0$ generically.

What if we move to a context with boundary?

Trace-Sobolev inequality

Theorem (Escobar'88 Indiana):

For $n \geq 3$, the trace-Sobolev inequality is:

$$\|\nabla\varphi\|^2_{L^2(\mathbb{R}^n_+)} \geq c(n)\|\varphi\|^2_{L^{2_*}(\partial\mathbb{R}^n_+)}, \quad \forall\varphi \in \dot{H}^1\left(\overline{\mathbb{R}^n_+}\right)$$

where

$$c(n) = (n - 2/2)^{1/2}\left(\sigma_{n-1}\right)^{\frac{1}{2(n-1)}}.$$

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The **extremal functions** are

$$\varphi(x,t) = \left(\frac{1}{(1+t)^2+|x|^2}\right)^{\frac{n-2}{2}}$$

Can we play the same game? (In a quantitative way)

Theorem (Ho'22 Proceedings of AMS):

There is a constant C>0, depending only on the dimension n, such that

$$\|\nabla\varphi\|^2_{L^2(\mathbb{R}^n_+)} - c(n)^2\|\varphi\|^2_{L^{2\frac{n-1}{n-2}}(\partial\mathbb{R}^n_+)} \geq C d\left(\varphi, \mathcal{M}_{\mathbb{R}^n_+}\right)^2 \quad \forall\varphi \in \dot{H}^1\left(\overline{\mathbb{R}^n_+}\right)$$

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$$d\left(\varphi, \mathcal{M}_{\mathbb{R}^n_+}\right) = \inf_{u\in\mathcal{M}_{\mathbb{R}^n_+}} \|\nabla(\varphi - u)\|_{L^2(\mathbb{R}^n_+)}$$

Conformal equivalence

- $\mathbb{R}^n_+$ and B^n are conformally equivalents;

There is a constant $C > 0$ such that

$$Q_{B^n}(\varphi) - Q(B^n, \partial B^n) \geq C d_{B^n}(\varphi, \mathcal{M}_{B^n})^2 \quad \forall \varphi \in H^1(B^n)$$

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What is the functional Q? The energy associated with the Escobar functional: Yamabe problem with boundary.

Yamabe problem with boundary

Let (M, g) be a manifold with boundary, $n \geq 3$. Can we find a metric $\tilde{g}$ conformal to g which is scalar-flat and has constant mean curvature on the boundary?

- Problem is reduced to find a positive solution $u > 0$ of

$$\begin{cases} 4\frac{(n-1)}{(n-2)}\Delta_g u - R_g u = 0 & \text{in } M \\ 4\frac{(n-1)}{(n-2)}\frac{\partial u}{\partial \nu} + 2(n-1)h_g u = 2(n-1)c u^{\frac{n}{n-2}} & \text{on } \partial M \end{cases}$$

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- ν is the exterior normal vector to ∂M and h_g is the mean curvature;
- The metric $\tilde{g} = u^{\frac{4}{n-2}} g$, is such that $R_{\tilde{g}} = 0$ and $h_{\tilde{g}} = c$.

Historically

- Functional associated:

$$Q(u) := \frac{4\frac{(n-1)}{(n-2)} \int_M |\nabla_g u|^2 \, dv + \int_M R_g u^2 dv + 2(n-1) \int_{\partial M} h_g u^2 d\sigma}{\left(\int_{\partial M} |u|^{2\frac{n-1}{n-2}} d\sigma\right)^{\frac{n-2}{n-1}}}.$$

- Escobar considered

$$Q(M, \partial M) := \inf \left\{ Q(\varphi) : \varphi \in C^1(\bar{M}), \varphi \neq 0 \text{ en } \partial M \right\}$$

which is a confomal invariant and showed that

$$Q(M, \partial M) \leq Q\left(B^n, \partial B^n\right),$$

where B^n is the unit ball.

Theorem (Escobar'92 Annals of Math.)

Let (M, g) be a compact manifold with boundary, $n \geq 3$. If

$$Q(M, \partial M) < Q\left(B^n, \partial B^n\right) \quad \text{and} \quad Q(M, \partial M) > -\infty$$

then there exists a smooth metric $\tilde{g} = u^{4/(n-2)} g$ that is scalar-flat and has constant mean curvature on the boundary.

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Contributions of:

- Escobar'92 Annals, Marques'05 Indiana, Marques'07 CAG, Almaraz'09 Pac. J. Math, Chen'09, Mayer-Ndiaye'17 JDG

Returning to stability...

Main Results

Theorem (Borquez, C., Van Den Bosch'25 submitted)

Let (M,g) be a manifold with boundary, $n \geq 3$, such that $Q(M,\partial M) < Q(B^n, \partial B^n)$ and $Q(M,\partial M) > -\infty$. Then, there exist a $C > 0$ and $\gamma \geq 0$, such that:

$$Q(\varphi) - Q(M,\partial M) \geq C d(\varphi,\mathcal{M})^{2+\gamma} \quad \forall \varphi \in H^1(M,\mathbb{R}_+),$$

where

$$d(\varphi,\mathcal{M}) = \frac{\inf_{u\in\mathcal{M}} \|\varphi - u\|_{H^1(M)}}{\|\varphi\|_{H^1(M)}}$$

Idea of the proof

- Define a suitable boundary functional $\tilde{Q}$ by means of an appropriate extension to the interior E:

$$\tilde{Q}(u) := Q(Eu) \quad u \in H^{1/2}(\partial M)$$

Here, $E : H^{1/2}(\partial M) \to H^1(M)$ satisfies:

$$\begin{cases} 4\frac{(n-1)}{(n-2)}\Delta_g Eu - R_g Eu = 0 & \text{in } M \\ Eu = u & \text{on } \partial M \end{cases}$$

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- To get a local stabulity result for $\tilde{Q}$: Łojasiewicz inequality;
  - Exponent γ is related to that!
- Obtain the stability for Q;

Many open questions...

- Generic stability;
- Examples where $\gamma > 0$
- Sharp quantitative stability;
- Other case of the Yamabe problem;

Thank you!
¡Muchas gracias!
Muito obrigada!