Taylor and Monomial Convergence

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with R. Ryan & N. Snigireva

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For each f in $\mathcal{F}(R)$ and $z \in R_n := R \cap \mathbb{C}^n$ we have

$$f(z) = \sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} a_\alpha z^\alpha$$

Monomial Expansion

$$\mathrm{mon}\mathcal{F}(R) = \left\{ z \in R : \sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} |a_\alpha z^\alpha| < \infty \quad \text{all } f \in \mathcal{F}(R) \right\}$$

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Could also consider

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$r(f, a)$ radius of convergence of $\sum_{k=0}^{\infty} P_k(z-a)$

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$$r(f,a) = \left(\limsup_{m\to\infty} \|P_m\|^{\frac{1}{m}} \right)^{-1}$$

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How are these two series expansions related? In particular, how are the two radii of convergence related?

Regular Holomorphic Functions

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$$\sup\left\{\rho : \sum_{k=0}^{\infty} |P_k|(z-a) \text{ converges on } B(a,\rho)\right\}$$

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Let $P\colon E \to \mathbb{K}$ to an m–homogeneous polynomial with associated symmetric m-linear mapping A.

Then

$$A(x_1, \dots, x_m) = \sum_{i_1} \cdots \sum_{i_m} x_{i_1} \dots x_{i_m} A(e_{i_1}, \dots, e_{i_m})$$

pointwise as an iterated limit

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Grecu & Ryan (2005)
This convergence is unconditional if and only if P is regular

Regular Holomorphic Functions

Theorem

B, Ryan & Snigireva (2025)

Let U be open in a Banach space E with an 1-unconditional Schauder basis. A function $f\colon U \to \mathbb{C}$ be holomorphic. Then f is regularly holomorphic if and only if, for every $a \in U$, the monomial expansion of f around a is absolutely convergent to f in some neighbourhood of a.

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Moreover, the radius of absolute convergence of the monomial expansion of f about a is equal to $|r|(f, a)$

Bohr Radius

Bohr (1914)

$\sum_{k=0}^{\infty} c_k z^k$ is a power series on the unit disc centred at 0, with

$$\left| \sum_{k=0}^{\infty} c_k z^k \right| \leq 1$$

for all z with $|z| < 1$ then

$$\sum_{k=0}^{\infty} |c_k z^k| \leq 1$$

for all z with $|z| \leq \frac{1}{3}$

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Moreover, $\frac{1}{3}$ is the optimal radius for which this inequality holds.

Bohr Radius

R a Reinhardt domain in $\mathbb{C}^n$ the Bohr radius of R, $K(R)$, is defined as the supremum over all $r \geq 0$ such that if

$$\sum_{\alpha} c_\alpha z^\alpha$$

is a power series on R with

$$\left| \sum_{\alpha} c_\alpha z^\alpha \right| \leq 1$$

for all z in R then

$$\sum_{\alpha} |c_\alpha z^\alpha| \leq 1$$

for all z in rR.

Bohr Radius

The works of Aizenberg, Boas, Boas and Kavinson, and Dineen and Timoney showed that there is a constant c, independent of $n \in \mathbb{N}$ and $1 \leq p \leq \infty$, such that

$$\frac{1}{c}\left(\frac{1}{n}\right)^{1-\frac{1}{\min(p,2)}} \leq K(B_{\ell_p^n}) \leq c\left(\frac{\log n}{n}\right)^{1-\frac{1}{\min(p,2)}}.$$

Homogeneous Bohr Radius

Defant, García & Maestre (2003)

Let $X = (\mathbb{C}^n, \|\cdot\|)$ with any norm for which the unit vector basis is a 1-unconditional Schauder basis, define $K_m(B_X)$ as the supremum over all r in $[0,1]$ such that if

$$\sum_{|\alpha|=m} c_\alpha z^\alpha$$

is an m-homogeneous polynomial on X with

$$\left| \sum_{|\alpha|=m} c_\alpha z^\alpha \right| \leq 1$$

for all z in B_X then

$$\sum_{|\alpha|=m} |c_\alpha z^\alpha| \leq 1$$

for all z in rB_X

Unconditional Basis Constant

E be a Banach space with an unconditional basis, $(x_n)_n$. The unconditional basis constant of $(x_n)_n$, $\chi((x_n)_n)$, is defined as the infimum of C such that

$$\left\|\sum_{k=1}^{\infty} \epsilon_k \mu_k x_k\right\| \leq C \left\|\sum_{k=1}^{\infty} \mu_k x_k\right\|$$

where $\mu_k \in \mathbb{C}$ and $|\epsilon_k| = 1$

Bohr radius and Unconditional Basis Constant

Let $X = (\mathbb{C}^n, \|\cdot\|)$ with any norm for which the unit vector basis is a 1-unconditional Schauder basis.
Then $\chi_{mon}(\mathcal{P}(^n X))$ denotes the unconditional basis constant of the monomials in $\mathcal{P}(^m X)$

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Lemma (Defant, García & Maestre) (2003)

$$K_m(B_X) = \frac{1}{\sqrt[m]{\chi_{mon}(\mathcal{P}(^m X))}}$$

Bohr Radius for Banach Lattice

Let $X = (\mathbb{C}^n, \|\cdot\|)$ with any norm for which the unit vector basis is a 1-unconditional Schauder basis then we may regard $\mathcal{P}(^m X)$ as a Banach lattice

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E be a complex Banach lattice and m be a positive integer. Define the m-th Bohr radius of E, $K_m(B_E)$ by

$$K_m(B_E) := \sup\{\rho : \sup_{z\in\rho B_E} ||P|(z)| \leq \|P\| \text{ for all } P \in \mathcal{P}(^m E)\}$$
$$= \sup\{\rho : \rho^m \|P\|_r \leq \|P\| \text{ for all } P \in \mathcal{P}(^m E)\}.$$

Radius of Regular Convergence and Bohr Radius

Theorem

B., Ryan & Snigireva (2025)

Let E be a complex Banach lattice and U an open subset of E. Let $f\colon U \to \mathbb{C}$ be a regular holomorphic function and $a \in U$. Then,

$$\liminf_{m\to\infty}(K_m(B_E))r(f,a) \leq |r|(f,a) \leq r(f,a).$$

Moreover, for each $a \in U$, both the upper and lower bounds are sharp.

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Moreover, for each $a \in U$, both the upper and lower bounds are sharp.

Corollary

If X is $(\mathbb{C}^n, \|\cdot\|)$ with any norm for which the unit vector basis is a 1-unconditional Schauder basis. Then

$$\frac{1}{\limsup_{m\to\infty} \sqrt[m]{\chi_{\mathrm{mon}}\mathcal{P}(^m X)}} r(f,a) \leq |r|(f,a) \leq r(f,a).$$

The Vanishing Lemma

X is the finite dimensional Banach lattice $(\mathbb{C}^n, \|\cdot\|)$, with any norm for which the unit vector basis is a 1-unconditional Schauder basis, then Defant, Galicer, Mansilla,Mastyło,Muro show that

$$\limsup_{m\to\infty} \sqrt[m]{\chi_{\mathrm{mon}}(\mathcal{P}(^m X))} = 1$$

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Theorem

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Let f be a holomorphic function on $(\mathbb{C}^n, \|\cdot\|)$, with any norm for which the unit vector basis is a 1-unconditional Schauder basis. Then $r(f,a) = |r|(f,a)$ for every a.

Real Analytic Functions on Finite Dimensions

This result is not true for real analytic functions on subsets of $\mathbb{R}^n$

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Hayman (1970) shows that if $f(x) = \sum_{k=0}^{\infty} P_k(x)$, where each P_k is a harmonic k-homogeneous polynomial, converges on the polydisc $\{(x_i)_{i=1}^n : |x_i| < r\}$ in ℓ_∞^n then $\sum_{k=0}^{\infty} |P_k|$ converges on the polydisc $\{(x_i)_{i=1}^n : |x_i| < r/\sqrt{2}\}$

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Moreover, an example is provided by Hayman shows that the factor of $r/\sqrt{2}$ is sharp

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Theorem

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Let $1 < p < \infty$. Then for each $\tau \in (0, 1)$ there is a holomorphic function f on B_{ℓ_p} with $r(f, 0) = 1$ and $|r|(f, 0) = \tau$

Radius of Regular Convergence on ℓ_1

Using a result of Matos (1988)

$$e^{-1} r(f, 0) \leq |r|(f, 0) \leq r(f, 0)$$

for every holomorphic function f on ℓ_1 or B_{ℓ_1}