

# SWAP RATE VARIANCE SWAPS

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## Abstract

We study the hedging and valuation of generalized variance swaps defined on a forward swap interest rate. Our motivation is the fundamental role of variance swaps in the transfer of variance risk, and the extensive empirical evidence documenting that the variance realized by interest rates is stochastic. We identify a hedging rule involving a static European contract and the gains of a dynamic position on forward interest rate swaps. Two distinguishing features arise in the context of interest rates: the nonlinear and multidimensional relationship between the values of the dynamically traded contracts and the underlying swap rate, and the possible stochasticity of the interest rate at which gains are reinvested. The combination of these two features leads to additional terms in the cumulative dynamic trading gains, which depend on realized variance and are taken into consideration in the determination of the appropriate static hedge. We characterize the static payoff function as the solution of an ordinary differential equation, and derive explicitly the associated dynamic strategy. We use daily interest rate data between 1997 and 2007 to test the effectiveness of our hedging methodology in arithmetic and geometric variance swaps and verify that the hedging error is small compared to the bid-ask spread in swaption prices.

# 1 Introduction

An extensive empirical literature has documented that the variance realized by interest rates is stochastic. Stochastic volatility as in Andersen and Lund (1997) and Ball and Torous (2000), random jumps as in Johannes (2004), switching regimes as in Gray (1996), are just a few examples of the statistical regularities observed in the behavior of interest rates that lead to stochastic realized variance. An investor that recognizes uncertainty about future realized variance as a risk factor distinct from the risk associated to the level of rates might want to modify his exposure with respect to variance risk. This is possible by trading in a simple contract that depends exclusively on variance risk: the variance swap. In this contract, two parties agree at  $t = 0$  to exchange a dollar amount proportional to the variance realized by a reference interest rate between 0 and  $T$ , against a fixed sum set at inception. The valuation problem associated to a variance swap is the computation of the fixed payment that makes the contract worthless at inception, therefore equivalent to valuing the floating leg.

In the absence of other liquidly traded instruments that could be used as hedges, the valuation of a variance swap must take into account subjective probabilities for outcomes in realized variance as well as preferences toward variance risk. However, it was shown independently by Neuberger (1992) and Dupire (1993) that the variance realized by a traded asset following a diffusion can be replicated as the difference between a static European contract on the asset and the cumulative gains that arise from a dynamic position in the same asset. Moreover, by Breeden and Litzenberger (1978) and Carr and Madan (1998) (see also Demeterfi et al., 1999), the payoff of the static contract can be replicated by a sufficiently rich portfolio of call and put European options. Strikingly, the replicating portfolio is derived under very mild assumptions about the volatility process of the traded asset. Therefore, the arbitrage-free price of realized variance is determined by the price of market observable European call and put options.

In reality, the existence of a rich set of liquidly traded options creates an alternative motivation for trading variance swaps. Future variance can be understood as a fundamental quantity simultaneously driving the prices of options and variance swaps, therefore a variance swap can

be used to set a speculative position by an investor who believes that option prices are far from his own subjective valuation. A variance swap provides the investor with a tool to speculate on future variance, *relative to the variance implied in observable option prices*, without taking any risk related to the direction of changes in the underlying.

Following the successful development of variance swaps defined on individual stocks and on equity indices, the literature has moved toward increasingly sophisticated computational approaches and more complex payoff structures. Little and Pant (2001) studied finite difference methods for variance swaps with discrete sampling in a local volatility model. Carr and Lewis (2004) explored the valuation and hedging of corridor variance swaps. The universe of contract payoffs has been expanded to include volatility swaps and volatility derivatives. Howison, Rafailidis and Rasmussen (2004) priced volatility derivatives in a partial differential equation framework with emphasis on an asymptotic analysis for fast mean reverting volatility. Javaheri, Wilmott and Haug (2004) studied volatility swaps in a GARCH setting and derived an approximate solution for the convexity correction between variance and volatility. Carr et al. (2005) priced options on realized variance under a pure jump process. Windcliff et al. (2006) focused on the hedging of discretely sampled volatility derivatives and the effects of jumps and hedging frequency. In Albanese et al. (2008) a general Markov model with jumps and stochastic volatility was considered and spectral methods were used to price derivatives on realized variance.

Carr and Lee (2008a) developed a robust and nonparametric replication methodology for volatility derivatives under an independence assumption between the underlying and its volatility. Broadie and Jain (2008) studied the pricing and risk management of volatility derivatives under Heston stochastic volatility model. The variance swap has also been used by Carr and Wu (2008) as a tool to extract market information about variance risk premia from observed option prices. Finally, in work done mutually independently from this paper, Carr and Lee (2008b) studied the hedging of variance options and obtained a replication result for the weighted variance of a general function of a positive, continuous semimartingale price process. As in our paper, they characterize the static hedging payoff as a solution of an ordinary differential

equation.

In this paper we investigate *generalized* variance swaps, in the context of interest rates, previously unexplored in the literature. We consider geometric and arithmetic variance swaps, as well as the wider class of weighted variance swaps, in the sense of Lee (2009), which includes corridor and gamma variance swaps. We focus on the explicit valuation and hedging of variance contracts defined on the volatility of interest rates derived from the Libor curve. This is the yield curve used in the dominant over-the-counter interest rate derivatives market. Brigo and Mercurio (2006) thoroughly cover commonly traded interest rate derivatives and related pricing models. As in earlier work in variance swaps, we identify a replication strategy formulated in terms of a static European contract and a dynamic trading rule. We design a hedging strategy that is consistent with current market practice by using the most flexible and liquidly traded at-the-money forward interest rate swaps and Libor deposits.

Two issues distinguish the hedging and valuation problem in the context of interest rates: First, the relationship between the underlying over which variance is computed (forward swap rate), and the values of available traded instruments (swaps, bonds, Libor deposits) is nonlinear and multidimensional. A forward swap rate is not a traded security. Second, the contract depends on the variance of interest rates. Therefore, assuming deterministic interest rates for reinvesting dynamically accrued gains, as is standard in the literature on variance swaps, is suspect. Instead, we choose to reinvest gains using Libor deposits at the prevailing (possibly stochastic) interest rate. We combine these two features under the assumption of a sufficiently flat forward curve, which allows us to transform the high dimensional hedging problem into a one dimensional computation, depending only on the dynamics of the forward swap rate over which variance is recorded. We consider generic dynamic trading strategies on forward swaps, and obtain a closed form representation for the cumulative gains, which include novel terms that depend on realized variance. Next, taking into account these novel terms, we proceed to identify a static hedging contract and a specific dynamic hedging rule that hedges the target variance swap. We characterize the static payoff function as the solution of an ordinary differential equation that we solve, up to quadrature, by means of a Green's function, and derive from it

a dynamic trading strategy. We perform this exercise explicitly for the hedging of the variance of increments in the underlying forward swap rate as well as for the variance of returns.

Absence of arbitrage implies that the initial value of the hedging portfolio is approximately equal to the present value of the floating leg of the variance swap, and exactly equal to it if there is no hedging error. In turn, the initial value of the hedging portfolio equals the initial value of the static hedge, as the dynamic hedging part is initially costless.

The effectiveness of the hedging strategy is tested empirically by comparing historically realized variance with the performance of the hedging portfolio, using historical interest rate data with daily frequency between 1997 and 2007. We find that the hedging methodology has small errors relative to the bid-ask spread of traded swaptions.

The paper is structured as follows. Section 2 describes the features of the variance swap contract. Section 3 reviews the valuation of a variance swap contract defined over a traded asset under reinvestment with zero rates. Section 4 presents the forward rate model, its connection with swap rates, and an approximation that allows us to reduce the high dimensionality of the yield curve to a single underlying and compute an expression for dynamic swap trading gains. In Section 5 we present the main theoretical result of the paper: an explicit hedging strategy based on a static European contract and a dynamic position in swaps. The static European payoff arises as the solution of an ordinary differential equation, solved through a Green's function approach. Section 6 contains empirical tests of the hedging methodology for arithmetic and geometric variance swaps. Our conclusions are in Section 7.

## 2 The Generalized Variance Swap Contract

A variance swap is a derivative security contingent on an underlying financial variable  $S$ . We model the underlying dynamics with a continuous time stochastic process  $S_t$ , with time measured in years. We assume that there are 252 trading days per year and, for an integer  $n > 0$ , we consider contracts with payoff at  $T = n/252$ . The variance swap is a contract between two parties who agree to exchange, at time  $T$ , the difference between a weighted integral of the variance realized historically by  $S_t$  on  $[0, T]$ , and a fixed amount  $T\beta^2$ . The final payoff is

$$V_T = \sum_{i=0}^{n-1} Q(S_{i\frac{T_0}{n}})(S_{(i+1)\frac{T_0}{n}} - S_{i\frac{T_0}{n}})^2 - T\beta^2, \quad (1)$$

for deterministic  $Q : \mathfrak{R} \rightarrow \mathfrak{R}$ . Some examples are:

- $Q(S_{i\frac{T_0}{n}}) = 1$ . Arithmetic variance swap.
- $Q(S_{i\frac{T_0}{n}}) = \frac{1}{(S_{i\frac{T_0}{n}})^2}$ . Geometric variance swap.
- $Q(S_{i\frac{T_0}{n}}) = \frac{1}{S_{i\frac{T_0}{n}}}$ . Gamma swap.
- $Q(S_{i\frac{T_0}{n}}) = 1_{S_{i\frac{T_0}{n}} \in [a,b]}$ . Corridor variance swap.

Process  $S_t$  could be the price of a traded asset, as in Section 3 in this paper, or an interest rate, as in Section 5. The most popular variance contract for stocks and equity indices is the geometric variance swap. This reflects the usual practice of understanding equity market dynamics in terms of lognormal distributions, as in the Black-Scholes model. In interest rates, which is our main focus, market practitioners often look at the market through the prism of normal distributions. Therefore, studying the valuation of arithmetic variance contracts is particularly relevant for interest rates. The gamma swap, as in Lee (2008), can be seen by its weighting function  $Q$  as an intermediate contract between the arithmetic and geometric variance swaps. The corridor variance swap is yet another extension, studied in Carr and Lewis (2004).

### 3 Valuation for a Traded Underlying

In order to illustrate the essence of the valuation methodology we review first the case in which the variance swap is defined over a traded asset and gains accumulated dynamically are reinvested at zero interest rate (the case of nonzero, deterministic interest rates can be solved essentially along the same lines). We consider the arithmetic variance case, because of its relevance in the interest rate market, and the geometric variance case, previously covered by Dupire (1993), Carr and Madan (1998), and Demeterfi et al. (1999). Following Carr and Wu

(2008), we assume a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  under the usual hypotheses. We assume that the traded asset, under the physical measure  $P$ , follows

$$\frac{dS_t}{S_{t-}} = \alpha_{t-} dt + \gamma_{t-} dW_t + \int_0^\infty (e^x - 1) [\mu(dx, dt) - \nu_t(x) dx dt], \quad (2)$$

in  $[0, T^*]$  for  $\alpha_t, \gamma_t$  and  $\nu_t$  predictable processes with respect to the filtration  $\mathcal{F}_t$  and  $W_t$  a  $P$ -Brownian motion. The random measure  $\mu(dx, dt)$ , with compensator  $\nu(t)$ , assigns a jump mark of size  $x$  if a jump happens at time  $t$ . We assume, as in Carr and Wu, that  $\int_0^\infty (|x| \wedge 1) \nu_t(x) < \infty$ . Finally, we assume that  $0 < S_t$ . Through stochasticity in  $\gamma$  this formulation includes standard stochastic volatility models with continuous paths. For our purposes, however, we do not need to identify the specific dynamics of  $\gamma$ .

We consider first the hedging and valuation of the floating leg of an arithmetic variance swap expiring at  $T_0 < T^*$ , (1) with  $Q = 1$ . This leg can be decomposed as

$$\begin{aligned} \sum_{i=0}^{n-1} (S_{(i+1)\frac{T_0}{n}} - S_{i\frac{T_0}{n}})^2 &= \sum_{i=0}^{n-1} S_{(i+1)\frac{T_0}{n}}^2 + S_{i\frac{T_0}{n}}^2 - 2S_{(i+1)\frac{T_0}{n}} S_{i\frac{T_0}{n}} \\ &= S_{T_0}^2 - S_0^2 - 2 \sum_{i=0}^{n-1} S_{i\frac{T_0}{n}} (S_{(i+1)\frac{T_0}{n}} - S_{i\frac{T_0}{n}}). \end{aligned} \quad (3)$$

Therefore, arithmetic variance realized between 0 and  $T_0$  can be replicated, through (3), by a hedging portfolio composed of a static contract with payoff

$$S_{T_0}^2 - S_0^2,$$

and the gains accrued by dynamically holding  $-2S_{i\frac{T_0}{n}}$  units of the underlying between  $t = i\frac{T_0}{n}$  and  $t = (i+1)\frac{T_0}{n}$ .

The present value of future realized variance must be equal to the present value of the replicating portfolio. In order to preclude arbitrage, we assume the existence of a risk neutral probability measure  $P^{RN}$  that makes discounted assets martingales. Because  $S_t$  is an asset price and we have assumed zero interest rates, it follows that  $S_t$  itself must be a martingale under the risk neutral measure. The initial value of the dynamic component of the hedging strategy is

$$-2E^{P^{RN}} \left[ \sum_{i=0}^{n-1} S_{i \frac{T_0}{n}} (S_{(i+1) \frac{T_0}{n}} - S_{i \frac{T_0}{n}}) | \mathcal{F}_0 \right] = 0, \quad (4)$$

by the martingale property of  $S_t$ . The static term in the hedging portfolio is a quadratic European payoff and can be priced explicitly by a replicating portfolio composed of European call and put options on  $S_{T_0}$ , as shown in Carr and Madan (1998). Therefore, the arbitrage-free value of future realized variance is determined by the price of liquidly traded European options. For simplicity, in this paper we will focus on the decomposition of realized swap rate variance up to a European payoff and a dynamic traded position, and will not be concerned with the decomposition of the European payoff into calls and puts.

A similar argument to that outlined above can be followed to identify a hedging portfolio for realized geometric variance. From Dupire (1993) and Demeterfi et al.(1999) we know that the optimal static hedging contract has a logarithmic shape with notional -2. The change in the value of the appropriate static contract for a fixed path can be expressed as a Taylor expansion over the discrete time grid implicit in (1) to obtain

$$\begin{aligned} -2 \ln(S_{T_0}) + 2 \ln(S_0) + \sum_{i=0}^{n-1} \frac{2}{S_{i \frac{T_0}{n}}} (S_{(i+1) \frac{T_0}{n}} - S_{i \frac{T_0}{n}}) &= \sum_{i=0}^{n-1} \frac{1}{S_{i \frac{T_0}{n}}^2} (S_{(i+1) \frac{T_0}{n}} - S_{i \frac{T_0}{n}})^2 \\ &\quad + O\left(\left(\frac{S_{(i+1) \frac{T_0}{n}} - S_{i \frac{T_0}{n}}}{S_{i \frac{T_0}{n}}}\right)^3\right) \end{aligned} \quad (5)$$

Therefore, realized geometric variance to be delivered at  $T_0$  is hedged in (5) in terms of a static logarithmic contract and a dynamic position on the asset to be rebalanced between 0 and  $T$  with notional  $\frac{2}{S_{i \frac{T_0}{n}}}$ . The last term in (5) is hedging error which, by analyticity of  $\ln(1+x)$  for  $|x| < 1$ , is of order three for small relative increments in the underlying process. It is in this sense that the hedging rule considered in (5) cancels exactly terms up to order two in the increments of the underlying process.

If jumps are absent in (2), regularity conditions on (2) (e.g. bounded  $\alpha$  and  $\gamma$ ), imply that the hedging error tends to zero in probability as the time interval between rehedging tends to

zero. In the limit we get

$$-2 \ln(S_{T_0}/S_0) + 2 \int_0^{T_0} \frac{1}{S_t} dS_t = \int_0^{T_0} \gamma_t^2 dt \quad (6)$$

A common feature of the two examples shown in this section is that the variance of interest is hedged through the gains of a suitable static payoff and those accrued by a dynamic position in the underlying. We will see that this structure is preserved in the hedging of swap rate variance, although for a more complicated static payoff function and dynamic trading rule.

## 4 Forward Curve Model

### 4.1 Swaps and Swap Rates

Swap rates are the most widely quoted interest rates for derivative transactions between high credit quality banks. Swap rate levels and their implied volatilities are readily extracted from very liquidly traded instruments. This is the motivation for defining contracts on the variance of a forward swap rate, as investigated in this paper. However, unlike in the examples explored in the previous section, forward swap rates are not traded assets. The most liquidly traded contracts defined on long dated Libor curve rates are swaps. And the value of a swap, or a forward swap, depends not only on the underlying swap rate, but on the yield curve up to the maturity of the swap.

As we intend to hedge variance contracts by trading in forward interest rate swaps, we must begin by specifying a general model of forward rates for a consistent description of forward swap rate dynamics and swap valuation.

Following Heath, Jarrow, and Morton (1994) we consider a family of instantaneous, continuously compounded forward rates  $f_t(u)$  for  $0 \leq t \leq u, u \leq T^*$ . The value of a zero coupon bond expiring at  $T \leq T^*$ , for  $t \leq T$ , is denoted by  $B_t(T)$ . Bond prices and forward rates are related through:

$$B_t(T) \equiv e^{-\int_t^T f_t(u) du}. \quad (7)$$

Investing 1 dollar at  $t$  in a Libor deposit that expires at  $T$  is equivalent to buying  $1/B_t(T)$  units of the zero coupon bond expiring at  $T$  and holding it until expiration.

Forward swap rates and related derivatives are associated to a discrete tenor structure—a finite set of dates with arbitrary start  $T_0$ ,  $0 \leq T_0 < T_1 < \dots < T_M < T^*$ , with  $T_{i+1} - T_i \equiv \delta$ . The fixed accrual period  $\delta$  is expressed as a fraction of a year; for instance,  $\delta = 1/2$  represents six months. Discrete forward Libor rates  $L(T_0), \dots, L(T_{M-1})$  associated to the tenor structure are defined from bond prices (7) as in Musiela and Rutkowski (1997) by setting

$$L_t(T_k) = \frac{1}{\delta} \left( \frac{B_t(T_k)}{B_t(T_{k+1})} - 1 \right) = \frac{1}{\delta} \left( e^{\int_{T_k}^{T_{k+1}} f_t(u) du} - 1 \right), \quad t \in [0, T_k], \quad k = 0, \dots, M-1. \quad (8)$$

Forward swaps are widely traded derivatives. A payer's swap holder makes fixed payments  $\delta K$  and receives floating payments  $\delta L_{T_i}(T_i)$  at  $T_{i+1}$ ,  $i = 0, \dots, M-1$ . The swap value at the beginning of the setting of the first floating payment is

$$V_{T_0}(T_0, T_M) = \delta \sum_{j=0}^{M-1} B_{T_0}(T_{j+1}) (S_{T_0}(T_0, T_M) - K),$$

where the forward swap rate  $S(T_0, T_M)$  at  $t$  is

$$S_t(T_0, T_M) \equiv \sum_{j=0}^{M-1} b_t(j) L_t(T_j) \quad \text{with} \quad b_t(j) \equiv \frac{B_t(T_{j+1})}{\sum_{i=0}^{M-1} B_t(T_{i+1})}. \quad (9)$$

The forward swap price at  $t < T_0$  is computed under the pricing measure  $P^{0,M}$  that uses

$$D_t(T_0, T_M) = \delta \sum_{j=0}^{M-1} B_t(T_{j+1}), \quad (10)$$

as numeraire, see Brigo and Mercurio (2006) for a detailed treatment. In this case we get

$$V_t(T_0, T_M) = \delta \sum_{j=0}^{M-1} B_t(T_{j+1}) \mathbf{E}^{P^{0,M}} [(S_{T_0}(T_0, T_M) - K) | \mathcal{F}_t]. \quad (11)$$

Furthermore, from the representation

$$S_t(T_0, T_M) = \frac{B_t(T_0) - B_t(T_M)}{\delta \sum_{j=0}^{M-1} B_t(T_{j+1})}, \quad (12)$$

and the form of the numeraire in (10), it is evident that  $S_t(T_0, T_M)$  is a discounted asset price, therefore a martingale under  $P^{0,M}$ . Therefore (11) becomes

$$V_t(T_0, T_M) = \delta \sum_{j=0}^{M-1} B_t(T_{j+1}) (S_t(T_0, T_M) - K) = D_t(T_0, T_M)(S_t(T_0, T_M) - k). \quad (13)$$

## 4.2 Flat Forward Curve Approximation

As presented so far, the shape and level of the forward curve is unconstrained. In this section we introduce a set of assumptions about the forward curve that facilitate the development of an analytical treatment of the variance swap.

**Assumption 4.1** *For  $t \leq T_0 \leq u \leq T_M$ , there exists  $\epsilon \ll 1$  such that*

$$(i) \quad 0 < f_t(u) < \epsilon,$$

$$(ii) \quad \max_{u \in [T_0, T_M]} f_t(u) - \min_{u \in [T_0, T_M]} f_t(u) < \epsilon^2.$$

Assumption 4.1 holds, for example, for  $\epsilon = 0.3$  if interest rates at all times and for all maturities are less than 30% per annum, and the maximum change in the magnitude of forward rates across the forward curve (for any fixed time  $t$ ) is less than 0.09 (or 9 % per annum). Both conditions have held for US dollar denominated interest rates of Libor quality for the period 1997-2007. Assumption 4.1 imposes interest rates much lower than 100 % per annum and a relatively flat forward curve. It is important to notice that Assumption (4.1) is static, and is postulated to hold at all times. We are not concerned here with the identification of the class of dynamic models that are consistent with Assumption (4.1).

From Assumption 4.1 and (8) it follows that for fixed  $t$  and  $j = 0, \dots, M - 1$ ,

$$\frac{1}{\delta} (e^{\delta \min_{u \in [T_0, T_M]} f_t(u)} - 1) \leq L_t(j) \leq \frac{1}{\delta} (e^{\delta \max_{u \in [T_0, T_M]} f_t(u)} - 1). \quad (14)$$

Since  $\delta < 1$ , and  $0 < \min_{u \in [T_0, T_M]} f_t(u) \leq \max_{u \in [T_0, T_M]} f_t(u) < \epsilon$ , it follows from an exact truncated Taylor expansion that

$$e^{\delta \min_{u \in [T_0, T_M]} f_t(u)} = 1 + \delta \min_{u \in [T_0, T_M]} f_t(u) + 1/2 K_1 (\delta \min_{u \in [T_0, T_M]} f_t(u))^2,$$

and

$$e^{\delta \max_{u \in [T_0, T_M]} f_t(u)} = 1 + \delta \max_{u \in [T_0, T_M]} f_t(u) + 1/2 K_2 (\delta \max_{u \in [T_0, T_M]} f_t(u))^2,$$

with unknown constants  $K_1, K_2 \in [0, 1]$ . Therefore

$$\min_{u \in [T_0, T_M]} f_t(u) \leq L_t(j) \leq \max_{u \in [T_0, T_M]} f_t(u) + 1/2 \delta (\max_{u \in [T_0, T_M]} f_t(u))^2. \quad (15)$$

The forward swap rate  $S_t(T_0, T_M)$  can be interpreted through (9) as a weighted average of forward Libor rates,  $L_t(T_0), \dots, L_t(T_{M-1})$ . Moreover, because bond prices are positive, weights defined as

$$\frac{B_t(T_{j+1})}{\sum_{i=0}^{M-1} B_t(T_{i+1})}, \quad j = 0, \dots, M-1,$$

are also positive and add up to one. It then follows immediately from (15) that

$$\min_{u \in [T_0, T_M]} f_t(u) \leq S_t(T_0, T_M) \leq \max_{u \in [T_0, T_M]} f_t(u) + 1/2 \delta (\max_{u \in [T_0, T_M]} f_t(u))^2. \quad (16)$$

Fixing  $\delta = 0.5$  at its standard value in the interest rate derivatives market, we use bound (16) to state an approximate relationship between the forward swap rate  $S_t(T_0, T_M)$  and the ratio of bond prices  $\frac{B_t(T_M)}{B_t(T_0)}$ .

**PROPOSITION 4.1** *If Assumption 4.1 holds, then the yield implied by a ratio of bonds,*

$$y_t(T_0, T_M) \equiv -\frac{1}{(T_M - T_0)} \ln\left(\frac{B_t(T_M)}{B_t(T_0)}\right)$$

*satisfies*

$$|y_t(T_0, T_M) - S_t(T_0, T_M)| \leq 5/4 \epsilon^2.$$

*Proof:*

$$\left| \frac{-1}{(T_M - T_0)} \ln\left(\frac{B_t(T_M)}{B_t(T_0)}\right) - S_t(T_0, T_M) \right| = \left| \frac{1}{(T_M - T_0)} \int_{T_0}^{T_M} f_t(u) du - S_t(T_0, T_M) \right|.$$

Invoking (16) we write

$$\begin{aligned} & \left| \frac{1}{(T_M - T_0)} \int_{T_0}^{T_M} f_t(u) du - S_t(T_0, T_M) \right| \leq \\ & \max \left\{ \left| \max_{u \in [T_0, T_M]} f_t(u) - \min_{u \in [T_0, T_M]} f_t(u) \right|, \right. \\ & \left. \left| \min_{u \in [T_0, T_M]} f_t(u) - (\max_{u \in [T_0, T_M]} f_t(u) + 1/2\delta(\max_{u \in [T_0, T_M]} f_t(u))^2) \right| \right\} \leq 5/4\epsilon^2, \end{aligned}$$

where the last inequality follows from Assumption 4.1. ■

It is in the sense of Proposition (4.1) that we write

$$\frac{B_t(T_M)}{B_t(T_0)} \approx e^{-(T_M - T_0)S_t(T_0, T_M)}. \quad (17)$$

### 4.3 Dynamic Swap Trading Gains

A generic dynamic hedging strategy involving forward interest rate swaps leads to gains that depend on the evolution of the underlying forward swap rate. In this section we compute such gains. Denote  $V_t^\tau$  for the value at  $t \leq T_0$  of a forward interest rate swap starting at  $T_0$  and ending at  $T_M$  with strike  $S_\tau$  set at  $\tau \leq t$ . In the expression for the swap value (13) the annuity factor (10) can be written as a ratio of bond prices and the forward swap rate using (12)

$$V_t^\tau = \frac{B_t(T_0) - B_t(T_M)}{S_t} (S_t - S_\tau). \quad (18)$$

By definition,  $V_\tau^\tau = 0$ .

Our hedging algorithm in Section 5 relies on a dynamic strategy executed between 0 and  $T_0$ , in which local swap gains are reinvested through Libor deposits up to  $T_0$ .

Let  $h : \mathfrak{R} \rightarrow \mathfrak{R}$  be deterministic and smooth. We write  $h_t$  for  $h(S_t)$ . For  $i = 0, \dots, n - 1$  consider a portfolio holding, between  $t = iT_0/n$  and  $t = (i + 1)T_0/n$ ,  $h_{iT_0/n}$  units of the forward

swap defined on the forward swap rate  $S_t$  with strike  $S_{iT_0/n}$ . Gains are collected at  $(i+1)T_0/n$  and reinvested in a Libor deposit of maturity  $T_0$ . Let  $G(h, n)$  be the value at  $T_0$  of gains accumulated by following this trading strategy for  $i = 0, \dots, n-1$ . We have

$$G(h, n) \equiv \sum_{i=0}^{n-1} h_{i \frac{T_0}{n}} (V_{(i+1) \frac{T_0}{n}}^{i \frac{T_0}{n}} - V_{i \frac{T_0}{n}}^{i \frac{T_0}{n}}) / B_{(i+1) \frac{T_0}{n}}(T_0), \quad (19)$$

therefore recalling that  $V_{i \frac{T_0}{n}}^{i \frac{T_0}{n}} = 0$ , and using (18), we get

$$G(h, n) \equiv \sum_{i=0}^{n-1} h_{i \frac{T_0}{n}} \frac{1 - B_{(i+1) \frac{T_0}{n}}(T_M) / B_{(i+1) \frac{T_0}{n}}(T_0)}{S_{(i+1) \frac{T_0}{n}}} (S_{(i+1) \frac{T_0}{n}} - S_{i \frac{T_0}{n}}). \quad (20)$$

Under Assumption 4.1, the ratios of bond prices in (20) can be replaced by (17) to define the approximate dynamic gain  $\hat{G}(h, n)$

$$\hat{G}(h, n) \equiv \sum_{i=0}^{n-1} h_{i \frac{T_0}{n}} \frac{1 - e^{-S_{(i+1) \frac{T_0}{n}}(T_M - T_0)}}{S_{(i+1) \frac{T_0}{n}}} (S_{(i+1) \frac{T_0}{n}} - S_{i \frac{T_0}{n}}). \quad (21)$$

Lighten the notation by introducing  $\epsilon_i \equiv S_{(i+1) \frac{T_0}{n}} - S_{i \frac{T_0}{n}}$ , for the increment in the underlying forward swap rate process and define  $L \equiv T_M - T_0$  to write

$$\hat{G}(h, n) = \sum_{i=0}^{n-1} h_{i \frac{T_0}{n}} \frac{1 - e^{-(S_{i \frac{T_0}{n}} + \epsilon_i)(T_M - T_0)}}{S_{i \frac{T_0}{n}} + \epsilon_i} \epsilon_i. \quad (22)$$

It is useful to express (22) as a sum of terms of increasing order in  $\epsilon_i$ . Because the ratio multiplying  $\epsilon_i$  is itself smooth in  $\epsilon_i$ , a Taylor expansion for small  $\epsilon_i$  leads to

$$\begin{aligned} \hat{G}(h, n) &= \sum_{i=0}^{n-1} h_{i \frac{T_0}{n}} \frac{1 - e^{-S_{i \frac{T_0}{n}} L}}{S_{i \frac{T_0}{n}}} \epsilon_i \\ &+ \sum_{i=0}^{n-1} h_{i \frac{T_0}{n}} \frac{-1 + (1 + S_{i \frac{T_0}{n}} L) e^{-S_{i \frac{T_0}{n}} L}}{S_{i \frac{T_0}{n}}^2} \epsilon_i^2 + R \end{aligned} \quad (23)$$

with

$$R = \sum_{i=0}^{n-1} \frac{h_{i \frac{T_0}{n}}}{2} \frac{2 + e^{-\eta_i L} (-\eta_i^2 L^2 - 2 - 2\eta_i L)}{\eta_i^3} \epsilon_i^3 \quad (24)$$

for some  $\eta_i$ ,  $S_{i\frac{T_0}{n}} \leq \eta_i \leq S_{(i+1)\frac{T_0}{n}}$ .

We have computed, in (23), the gains that arise from dynamically trading in forward swaps with notional  $h(S_t)$  and reinvesting through Libor deposits at the prevailing Libor rate. This is used in the next section to identify the hedging portfolio for generalized variance swaps.

## 5 Hedging Swap Rate Variance

### 5.1 Structure of the Hedging Problem

In this section we identify a hedging portfolio for a generalized variance swap linked to realized swap rate variance. As in Section 3, the essence of the method consists on hedging weighted realized variance as the difference between a suitable static payoff and the gains that arise from a dynamic position. We depart from Section 3 in the fact that the underlying forward swap rate is not directly tradeable, and by reinvesting dynamic gains at the, possibly stochastic, rate implied by  $B_t(T_0)$ .

We consider a weighted variance swap defined on the variance realized between 0 and  $T_0$  by a forward swap rate. This rate corresponds to an interest rate swap defined on a discrete tenor structure as in Section 4, beginning at  $T_0$  and ending at  $T_M < T^*$ . The starting and ending dates  $T_0$  and  $T_M$ , associated with the swap rate, *are fixed* for all  $t \leq T_0$ . We write  $S_t$  for  $S_t(T_0, T_M)$ . Market participants are typically interested in realized variance for relatively short term horizons, defined on forward swap rates of commonly quoted length. For example, to fix ideas, we might take  $T_0 = 0.5$  (6 months) and  $T_M - T_0 = 10$  (10 years).

In reality, market participants observe relatively few liquid Libor and spot swap rates (3m, 6m, 1y, 2y, 3y, 5y, 10y, 20y, 30y) from which forward swap rates are computed through an interpolation mechanism that varies across market participants. Furthermore, as the variance swap ages, the starting date of the underlying interest rate swap becomes non standard, and the relevant underlying forward swap rate not directly observable. We are assuming that the counterparties in the variance contract agree on method for the daily computation of forward swap rates, or that they agree on a third party provider (e.g. Bloomberg).

As in Section 3 we model the dynamics of the underlying forward swap rate on a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  under the usual hypotheses, and postulate that the forward swap rate  $S_t$ , under the physical measure  $P$ , follows

$$\frac{dS_t}{S_{t-}} = \alpha_{t-} dt + \gamma_{t-} dW_t + \int_0^\infty (e^x - 1)[\mu(dx, dt) - \nu_t(x) dx dt], \quad (25)$$

under the technical conditions imposed for (2) in Section 3.

The floating leg of the contract we intend to hedge is

$$\sum_{i=0}^{n-1} Q(S_{i\frac{T_0}{n}}) \epsilon_i^2, \quad (26)$$

for a deterministic  $Q : \mathfrak{R} \rightarrow \mathfrak{R}$  and  $\epsilon_i = S_{(i+1)\frac{T_0}{n}} - S_{i\frac{T_0}{n}}$ . Some possible choices of  $Q$  were introduced in Section 2.

Let  $f : \mathfrak{R} \rightarrow \mathfrak{R}$  be a smooth function, to be interpreted as the payoff of a European contract when acting on  $S_{T_0}$ . We write  $f_t$  for  $f(S_t)$ ,  $t \leq T_0$ . We are looking for a hedging rule, defined by a static contract with payoff  $f$  and a dynamic hedging rule with notional  $h$  as in (23), such that the wealth delivered by the hedging portfolio at  $T_0$

$$f_{T_0} - f_0 + \hat{G}(h, n), \quad (27)$$

is an effective hedge for (26).

Smoothness of  $f$  allows us to write

$$f_{T_0} - f_0 = \sum_{i=0}^{n-1} f_{(i+1)\frac{T_0}{n}} - f_{i\frac{T_0}{n}} = \sum_{i=0}^{n-1} f'_{i\frac{T_0}{n}} \epsilon_i + \frac{1}{2} f''_{i\frac{T_0}{n}} \epsilon_i^2 + \frac{1}{6} f'''(\xi_i) \epsilon_i^3. \quad (28)$$

for some  $\xi_i$ ,  $S_{i\frac{T_0}{n}} \leq \xi_i \leq S_{(i+1)\frac{T_0}{n}}$ . Therefore, the total wealth delivered by the hedging portfolio at  $T_0$  is the sum of (23) and (28). Next, we must choose  $h, f$  so that the terminal value of the hedging portfolio approximates (26). We proceed sequentially, in increasing powers of  $\epsilon_i$ .

## 5.2 Hedging First Order Terms

The payoff of the generalized variance swap we are considering has no terms linear in  $\epsilon_i$ , therefore the net linear contribution of the hedging portfolio must be vanish. From (23) and (28), this is equivalent to imposing

$$\sum_{i=0}^{n-1} h_{i, \frac{T_0}{n}} \frac{1 - e^{-S_{i, \frac{T_0}{n}} L}}{S_{i, \frac{T_0}{n}}} \epsilon_i + \sum_{i=0}^{n-1} f'_{i, \frac{T_0}{n}} \epsilon_i = 0, \quad (29)$$

therefore

$$h_{i, \frac{T_0}{n}} = - \frac{f'_{i, \frac{T_0}{n}} S_{i, \frac{T_0}{n}}}{1 - e^{-S_{i, \frac{T_0}{n}} L}}. \quad (30)$$

Notional (30) is more easily interpreted recalling (17) and (18) to recognize that

$$h_{i, \frac{T_0}{n}} \approx - \frac{f'_{i, \frac{T_0}{n}} B_{i, \frac{T_0}{n}}(T_0)}{D_{i, \frac{T_0}{n}}}.$$

The bond price in the numerator of the notional function arises from the fact that gains are reinvested as Libor deposits, and the denominator is a consequence of the fact that the position involves swaps.

## 5.3 Hedging Second Order Terms

Next, we match second order terms. On the hedging side, the contribution from (28) must be added to the contribution from (23), which is rewritten in terms of  $f$  using (30) and some algebra. These must hedge the second order term in the weighed variance swap:

$$\frac{1}{2} f''_{i, \frac{T_0}{n}} + f'_{i, \frac{T_0}{n}} \left( \frac{1}{S_{i, \frac{T_0}{n}}} - \frac{L e^{-L S_{i, \frac{T_0}{n}}}}{1 - e^{-L S_{i, \frac{T_0}{n}}}} \right) = Q(S_{i, \frac{T_0}{n}}). \quad (31)$$

As this must hold for any random value of  $S_{i, \frac{T_0}{n}} \in (0, +\infty)$ , we look for  $f = f(x)$  that satisfies

$$\frac{1}{2} f'' + f' \left( \frac{1}{x} - \frac{L e^{-xL}}{1 - e^{-xL}} \right) = Q(x). \quad (32)$$

We choose, for convenience, to impose boundary conditions on the level and slope of  $f$  at  $x = S_0$ ,

$$\{f(S_0), f'(S_0)\}. \quad (33)$$

The absence of terms proportional to  $f$  in (32) implies that (32) can be written as a system of two ordinary differential equations of first order. These can be solved sequentially with a standard numerical integrator with very high precision and ease of implementation. It is also possible to solve (32) through a Green's function approach, as shown below. For the case of arithmetic and geometric variance we have compared the corresponding static hedging payoffs that arise from solving (32) and confirmed that both computational approaches lead to undistinguishable solutions in economic sense.

In order to develop a Green's function approach to the solution of (32) with boundary conditions (33) we begin by expressing the solution of interest as

$$f = f^h + f^i \quad (34)$$

where  $f^h$  is a solution to the homogeneous differential equation associated to (32) (setting  $Q = 0$ ) with boundary conditions (33), and  $f^i$  a particular solution to (32) with homogeneous boundary conditions ( $f^i(S_0) = 0, (f^i)'(S_0) = 0$ ). This decomposition guarantees that  $f$  solves (32) while satisfying (33).

The homogeneous problem is defined by

$$(f^h)'' + (f^h)' \left( \frac{2}{x} - \frac{2Le^{-xL}}{1 - e^{-xL}} \right) = 0. \quad (35)$$

For the general solution to (35) we define an integrating factor

$$I \equiv e^{\int_0^x \left( \frac{2}{u} - \frac{2Le^{-uL}}{1 - e^{-uL}} \right) du},$$

and multiply it by (35) to obtain

$$\frac{d}{dx}[(f^h)' e^{\int_0^x (\frac{2}{u} - \frac{2Le^{-uL}}{1-e^{-uL}}) du}] = 0,$$

and therefore,

$$(f^h)' = C e^{\int_0^x (\frac{-2}{u} + \frac{2Le^{-uL}}{1-e^{-uL}}) du} = C \frac{x^2}{(1 - e^{-xL})^2},$$

for some constant  $C$ . Therefore, the general solution to (35) is

$$f^h(x) = A^h + B^h \phi(x), \quad (36)$$

with

$$\phi(x) \equiv \int_0^x \frac{u^2}{(1 - e^{-uL})^2} du, \quad (37)$$

where the integrand in (37) is smooth and with finite limit as  $x \rightarrow 0$  therefore computed numerically with arbitrarily high precision. Constants  $A^h, B^h$  are determined by fixing the value and slope of  $f^h$  at  $x = S_0$  in accordance with (33).

The second part to the solution of (32),  $f^i$ , is a particular solution to (32) but with zero boundary conditions. It can be represented as

$$f^i(x) = \int_0^{+\infty} G(x, u) Q(u) du \quad (38)$$

where the  $G(x, u)$  is the Green's function that solves

$$\frac{1}{2} \frac{\partial G^2}{\partial x^2} + \frac{\partial G}{\partial x} \left( \frac{1}{x} - \frac{Le^{-xL}}{1 - e^{-xL}} \right) = \delta(x - u). \quad (39)$$

The Green's function  $G(x, u)$  is constructed by pasting two solutions of the homogeneous equation at  $x = u$  and setting appropriate boundary conditions. A standard reference is Stakgold (1997). We get

$$\begin{aligned} G(x, u) &= A + B\phi(x) \text{ for } x \leq u, \\ G(x, u) &= C + D\phi(x) \text{ for } x > u. \end{aligned} \quad (40)$$

Boundary conditions are zero at  $x = S_0$ . Therefore we have two cases:

If  $S_0 \leq u$ , then  $A + B\phi(S_0) = 0$  and  $B\phi'(S_0) = 0$ , implying  $A = B = 0$ . At  $x = u$  the Green's function must be continuous and have a jump of magnitude 1 in its derivative with respect to  $x$ , therefore  $0 = C + D\phi(u)$  and  $1 = D\phi'(u)$ , implying  $C = -\phi(u)/\phi'(u)$  and  $D = 1/\phi'(u)$ .

Figure 1 shows the Green's function that solves (39) for the case of  $S_0 \leq u$ . We also show in this Figure the Green's function for the traded asset case of Section 3, which solves  $\frac{1}{2} \frac{\partial G^2}{\partial x^2} = \delta(x - u)$ .

If  $S_0 < u$ , then  $C + D\phi(S_0) = 0$  and  $D\phi'(S_0) = 0$ , implying  $C = D = 0$ . At  $x = u$  the Green's function must be continuous and have a jump of magnitude 1 in its derivative with respect to  $x$ , therefore  $0 = A + B\phi(u)$  and  $1 = -B\phi'(u)$ , implying  $A = \phi(u)/\phi'(u)$  and  $B = -1/\phi'(u)$ .

Figure 2 shows the Green's function that solves (39) for the case of  $S_0 > u$ . We also show in this Figure the Green's function for the traded asset case of Section 3, which solves  $\frac{1}{2} \frac{\partial G^2}{\partial x^2} = \delta(x - u)$ .

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## 5.4 Hedging Error

By matching first and second order terms through our choice of  $h$  and  $f$  we have exploited all the degrees of freedom available to us in the composition of the hedging portfolio (27). The remaining part is hedging error, and fully explained by the third order terms implied by our choice of  $h$  and  $f$ . We collect them from (28), and (24) using (30). The hedging error for a generalized variance swap defined by (32) is given by

$$\sum_{i=0}^{n-1} \left[ -\frac{f'_i \frac{T_0}{n} S_i \frac{T_0}{n}}{2(1 - e^{-S_i \frac{T_0}{n} L})} \frac{2 + e^{-\eta_i L} (-\eta_i^2 L^2 - 2 - 2\eta_i L)}{\eta_i^3} + \frac{1}{6} f'''(\xi_i) \right] \epsilon_i^3. \quad (41)$$

for  $S_i \frac{T_0}{n} \leq \eta_i \leq S_{(i+1)} \frac{T_0}{n}$ ,  $S_i \frac{T_0}{n} \leq \xi_i \leq S_{(i+1)} \frac{T_0}{n}$  and  $f$  solving (32) with boundary conditions (33).

## 5.5 Hedging Arithmetic and Geometric Variance

The arithmetic variance swap problem corresponds to taking  $Q(x) = 1$  in (32). Boundary conditions for the static contract used to hedge arithmetic variance are chosen to be

$$f(S_0) = f'(S_0) = 0.$$

With these boundary conditions, the solution to (32) agrees at  $x = S_0$ , in level and slope, with the payoff function used in Section 3 for the arithmetic variance in the traded underlying case,  $f(x) = (x - S_0)^2$ . Moreover,  $f^h$  in (34) is, with these boundary conditions, exactly zero, therefore the solution to (32) is computed from the Green's function (40) as in (38). Figure 3 shows  $f$ , for  $S_0 = 0.06$  and  $L = 10$  (10 years). Figure 4 shows how this solution differs from the traded asset case payoff function  $f(x) = (x - S_0)^2$ .

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The geometric variance swap problem corresponds to taking  $Q(x) = \frac{1}{x^2}$  in (32). The boundary conditions for the static hedge in the geometric variance case are

$$f(S_0) = 0 \quad \text{and} \quad f'(S_0) = -2/S_0,$$

which are used to identify  $A^h$  and  $B^h$  in (36). In this case, the boundary conditions are chosen for our solution to match the level and slope of the static payoff function of the hedging of geometric variance in the traded asset case,  $f(x) = -2\ln(x/S_0)$ , at  $x = S_0$ . The inhomogeneous contribution to  $f$  in (34) is computed from the Green's function as in (38).

Figure 5 shows the solution to (32) for  $L=10$ , and Figure 6 shows how this solution differs from  $f(x) = -2\ln(x/S_0)$  for  $S_0 = 0.06$ .

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## 5.6 Implementation and Valuation

In summary, the hedging strategy for generalized variance swaps consists on holding a European contract with static payoff  $f$  that solves (32) and to execute, for the corresponding  $f'$ , a dynamic trading strategy holding  $h_t = \frac{f'_t S_t}{1 - e^{-S_t(T_M - T_0)}}$  at-the-money forward swaps and reinvesting instantaneous dynamic gains via Libor deposits. The payoff function  $f$  and its derivative are computed only at the beginning of the variance swap and stored as a table for later use.

This is a hedging strategy derived under Assumption 4.1. The execution of the strategy in reality involves investing in interest rate swaps and Libor deposits that are priced in the market without any approximation. Therefore an alternative dynamic hedging rule consists on using

$$\frac{f'_t B_t(T_0)}{D_t} \quad \text{instead of} \quad \frac{f'_t S_t}{1 - e^{-(T_M - T_0)S_t}}$$

as the notional of the position in swaps. Notice that both quantities can be computed directly from the observable yield curve at  $t$ .

The present value of the floating leg of the generalized variance swap is approximately equal to the present value of the hedging portfolio. This is approximate because of the existence of third order hedging errors. And the present value of the hedging portfolio is that of the static European payoff, because the dynamic portfolio is initially worthless.

## 6 Empirical Testing

In this section we test empirically the accuracy of the hedging strategy outlined in Section 5 for arithmetic and geometric variance contracts. Hedging errors arise from third order effects in the increments of the underlying process due to discrete hedging and jumps in the dynamics. We also test, implicitly, the appropriateness of the flat curve approximation (17) used in (21).

### 6.1 The Data

We use daily interest rate data from December 1, 1997 to December 1, 2007, provided by Lehman Brothers. For each day in the sample, the set of rates is composed of: 3 month Libor, the first eight EuroDollar futures, and liquidly traded swap rates with tenors 2y, 3y, 5y, 10y,

15y, 20y and 30y. Rate levels are interpolated to construct a continuous instantaneous forward rate curve. Zero coupon bond prices, annuity factors, and forward swap rates are priced exactly from the continuous interest rate curve.

## 6.2 Testing Strategy

We test the accuracy of the hedging strategy implied by (30) when  $f$  is the solution of (32) with the  $Q$  that corresponds to the relevant variance swap. We use daily frequency for recording variance increments and rebalancing the dynamic portfolio. For convenience, we adopt a change of notation to measure time in days. We define dollar hedging error as the difference between hedging gains and realized variance

$$\text{dollar error} \equiv f_N - f_0 - \sum_{i=0}^{N-1} \frac{f'_i B_i(T_0)}{D_i} (V_{i+1}^i - V_i^i) - \sum_{i=0}^{N-1} Q(S_i)(S_{i+1} - S_i)^2. \quad (42)$$

The payoff function  $f$  and its derivative are computed only once per variance swap and stored as a table. Then, having observed the full historical realization of forward curves for days  $i = 0, \dots, N$  we compute, for each day, the bond price  $B_i(T_0)$ , annuity factor  $D_i$ , the exact gain of the swap initiated the previous day, and the change in the forward swap rate over consecutive days. Then, the dollar hedging error in (42) can be computed exactly. Because market participants are used to quote prices in terms of implied volatilities, we compute (42) and convert it to volatility terms for displaying purposes.

We test the hedging of variance swaps of various lengths and underlying rates. The variance swaps we consider cover non overlapping intervals for the recording of variance. Therefore, in 10 years of data we have 40 observations for variance swaps of length 3 months, and 10 observations for length 1 year.

## 6.3 Results

Figure 7 displays realized arithmetic variance against the gain delivered by the hedging strategy for a variance swap of length 3 months, defined on a forward swap rate of length 5 years. Therefore, the underlying rate at the inception of the contract is the 3m5y forward swap rate.

As the variance contract ages, the underlying rate becomes a forward swap rate with 5y tenor but shorter starting date. Each point in the plot is an observation, corresponding to a 3 month interval. Variances are multiplied by 4 to be annualized and by  $10^4$  to be expressed in basis points. Moreover, we are hedging variance, but choose to display the square root of hedge gain against the square root of realized variance to be able to show volatility units. The square root of realized variance in Figure 7 ranges from 40 to 150 basis points per year, reflecting a wide dispersion in realized variance. The fact that all points are aligned near the line of slope 1 is evidence of the small hedging error relative to the stochasticity of realized arithmetic variance.

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This size of the typical hedging error is reflected more precisely in Figure 8, showing the histogram of hedging errors (also in volatility units). The bid-ask spread of swaptions, which are the most liquid call and put options on swap rates, was between 1 and 5 basis points in volatility units in 2007, depending on the moneyness of the option, and higher in earlier years. Because the replication of a static European contract in terms of calls and puts as in Carr and Madan (1998) involves options of all strikes, and considering the relatively exotic nature of the variance swap, we can safely assume that 3 basis points in volatility is a conservatively small bid-ask spread for the price of a variance swap. The hedging errors in Figure 8 are much smaller than this, suggesting that the hedging methodology is sufficiently accurate.

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Figure 10, now at the end of the document, should be inserted here.

The importance of using a static payoff function that accounts for the additional variance related dynamic gains discussed in Section 5 is tested in Figure 9 where we repeat the discrete hedging test described above, but using  $f(x) = (x - S_0)^2$  (both as static payoff and in determining  $f'$  that goes into the dynamic notional). Hedging errors in Figure 9 are one order of magnitude bigger than in Figure 8 and close to the natural bound suggested by the bid-ask

spread of swaptions. We also test the dependence of the quality of the hedging methodology on the characteristics of the variance swap. Figure 10 shows hedging errors for a 1 year variance swap defined on a 10 year swap rate (initially 1y10y forward swap rate). A longer variance swap implies less non overlapping periods, hence less observation points. Errors are comparable in magnitude to those in Figure 8.

We also test the hedging of geometric variance for a 3 month variance swap on the 5 year swap rate. Figure 11 shows the square root of the gain of the hedging portfolio against the square root of realized geometric variance. We are normalizing variance in agreement with market conventions to obtain annualized Black volatilities. The wide dispersion of values for realized variance in Figure 11 suggests strong stochasticity for geometric realized variance.

A typical basis point volatility of 100 basis points per year, and interest rates at 5%, imply that the previously assumed bid-ask spread of 3 volatility basis points for the variance swap is equivalent to a 0.6 Black volatility bid-ask spread. Volatility hedging errors in Figure 12, for the 3m5y variance swap, and in Figure 13 for the 1y10y variance swap, are smaller than this, suggesting that the method is effective in hedging geometric variance. Tests for hedging geometric variance using the logarithmic function of the traded asset case from Section 3 lead to hedging errors (not shown) that are significantly worse than those that arise from using the solution of (32).

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Figure 12, now at the end of the document, should be inserted here.

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## 7 Conclusions and Extensions

We have developed a hedging and valuation methodology for generalized variance swaps defined on a forward swap interest rate. The class of contracts under study includes arithmetic and geometric variance swaps as well as corridor and gamma swaps. The method uses a static payoff, represented as the solution of an ordinary differential equation, and a dynamic trading rule

implemented explicitly in terms of the most liquidly traded instruments according to current practice in the fixed income markets. Empirical results for arithmetic and geometric variance show that the hedging methodology is effective, leading to hedging errors smaller than the relevant bid-ask spreads. The accuracy of the hedging strategy shows that valuing the floating leg of the variance swap as the present value of the hedging portfolio is a very good approximation. This, in turn, is equal to the initial value of the static contract, as the dynamic hedging rule is initially worthless.

The valuation of volatility swaps and other non linear payoffs in realized variance, still in the context of interest rates, remains an open question. The nonlinear relationship between underlying swap rates and traded swaps, and the stochasticity of reinvestment rates, which are the main novel issues considered in this paper, appear also in the hedging of non linear swap rate variance derivatives. Preliminary work in this direction is in Merener (2009), where the robust and non parametric technology developed by Carr and Lee (2008a) for volatility swaps on a traded asset under zero interest rates, is adapted to the interest rate setting. This is accomplished, in an approximate manner, by merging some of the simplest results in Carr and Lee (2008a) with convexity corrections in the swap rate dynamics that guarantee absence of arbitrage. Exploiting this connection exhaustively remains for future work.

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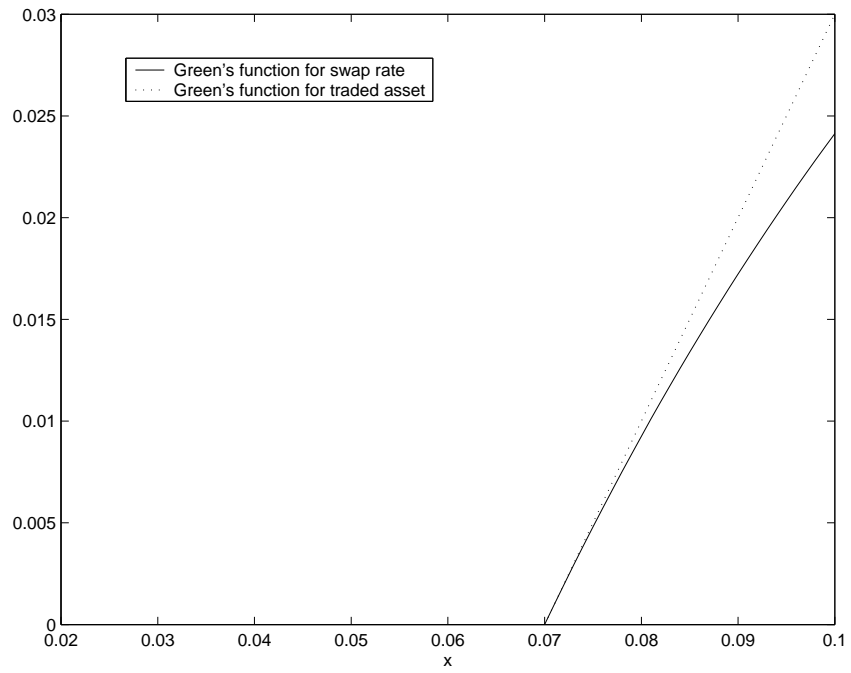


Figure 1: Green's functions for  $u=0.07$  and  $S_0 = 0.06$ , for swap rate underlying and traded asset underlying.

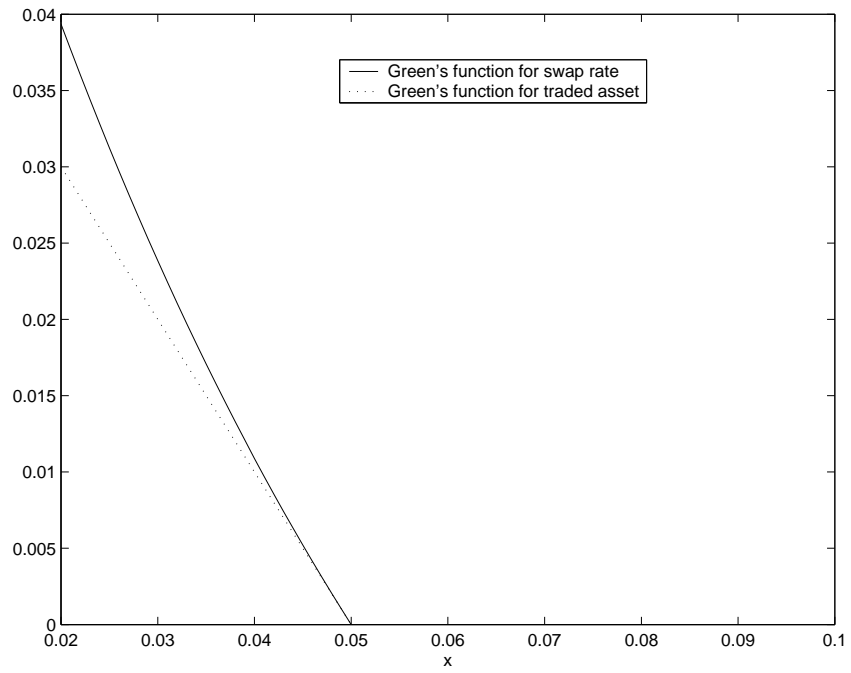


Figure 2: Green's functions for  $u=0.05$  and  $S_0 = 0.06$ , for swap rate underlying and traded asset underlying.

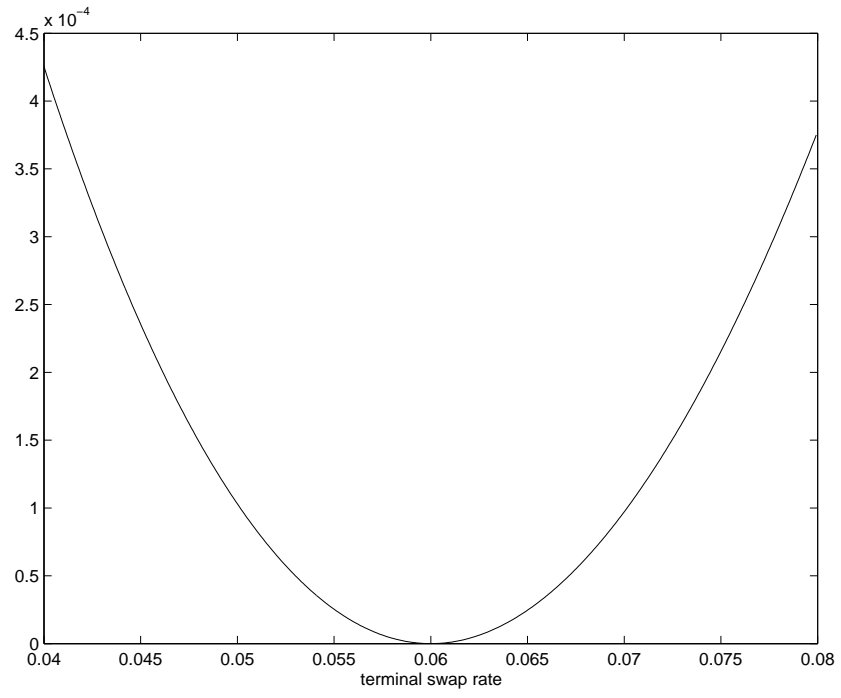


Figure 3: Static payoff function for hedging arithmetic variance.

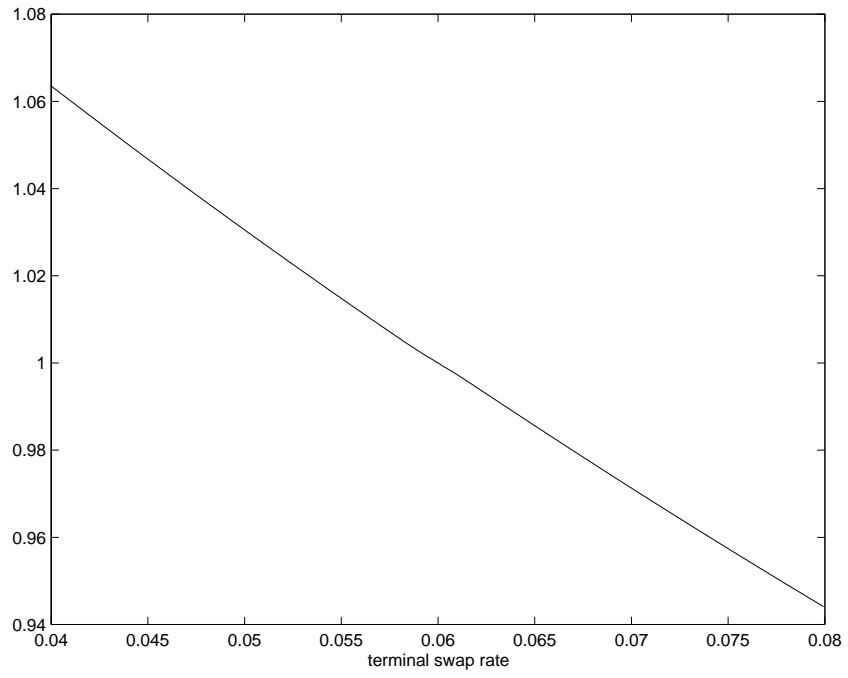


Figure 4: Ratio of payoff function for arithmetic variance and  $f(x) = (x - S_0)^2$

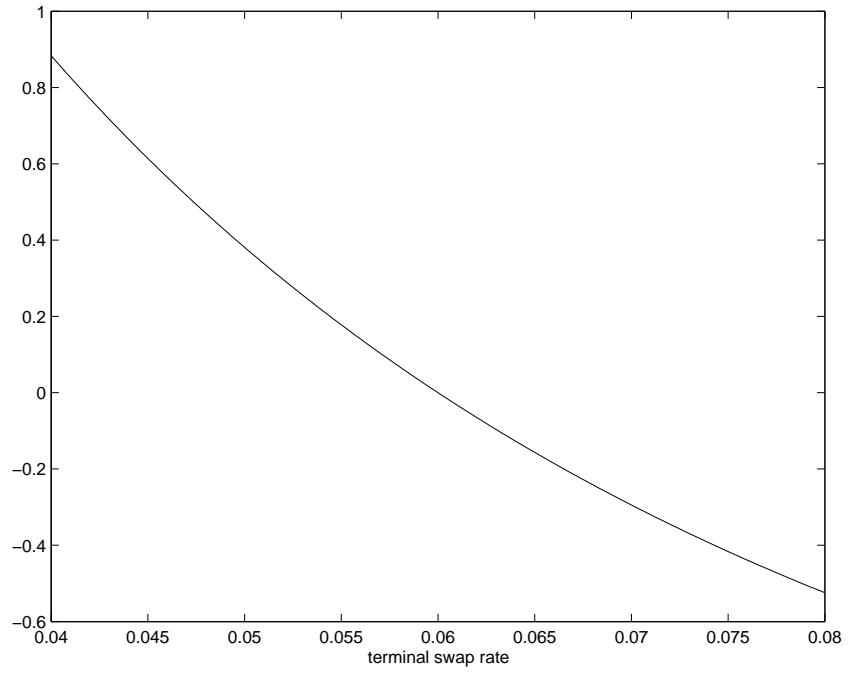


Figure 5: Static payoff function for hedging geometric variance.

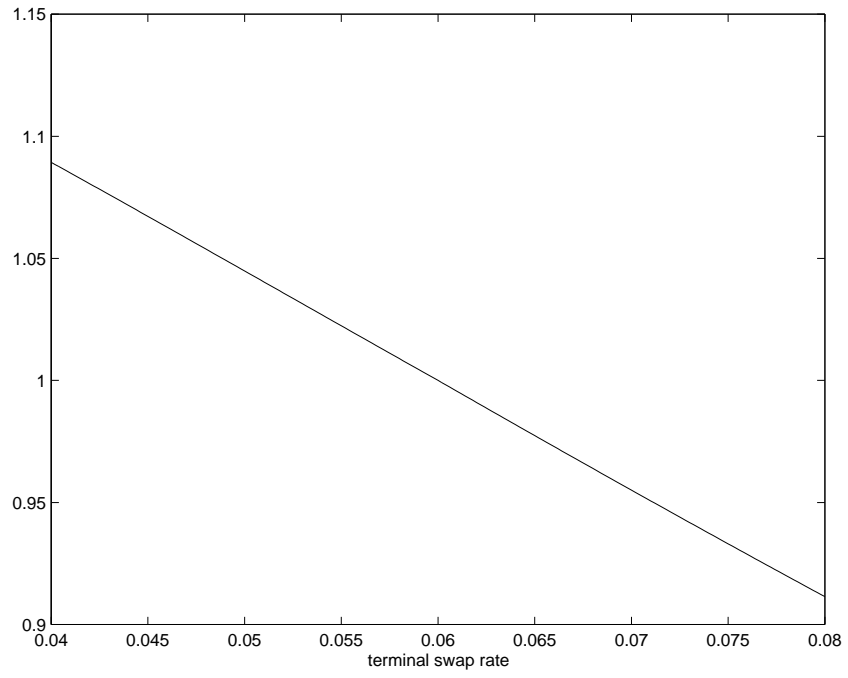


Figure 6: Ratio of payoff function for hedging geometric variance and  $f(x) = -2\ln(x/S_0)$

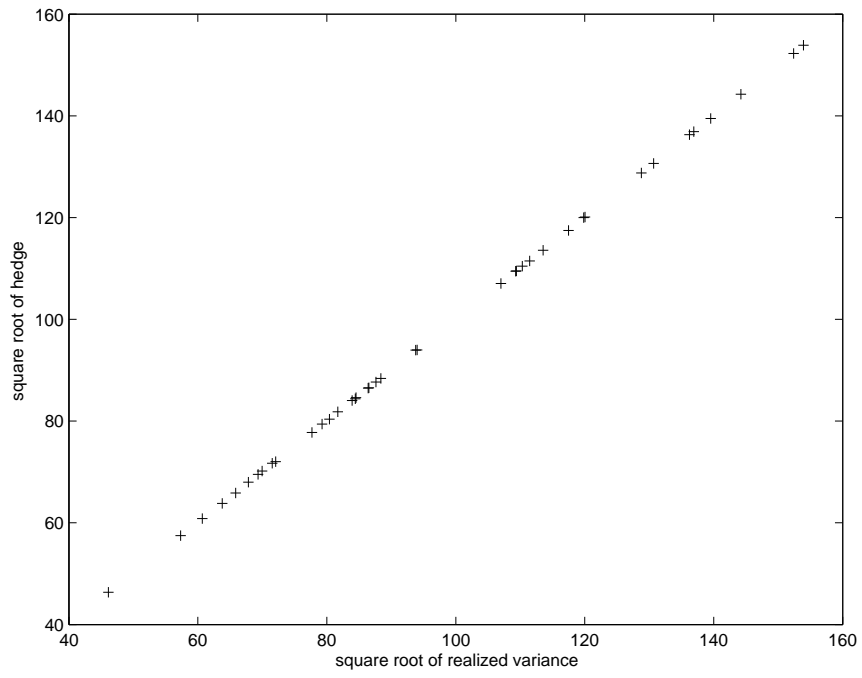


Figure 7: Realized arithmetic variance vs. hedge gain, 3m5y

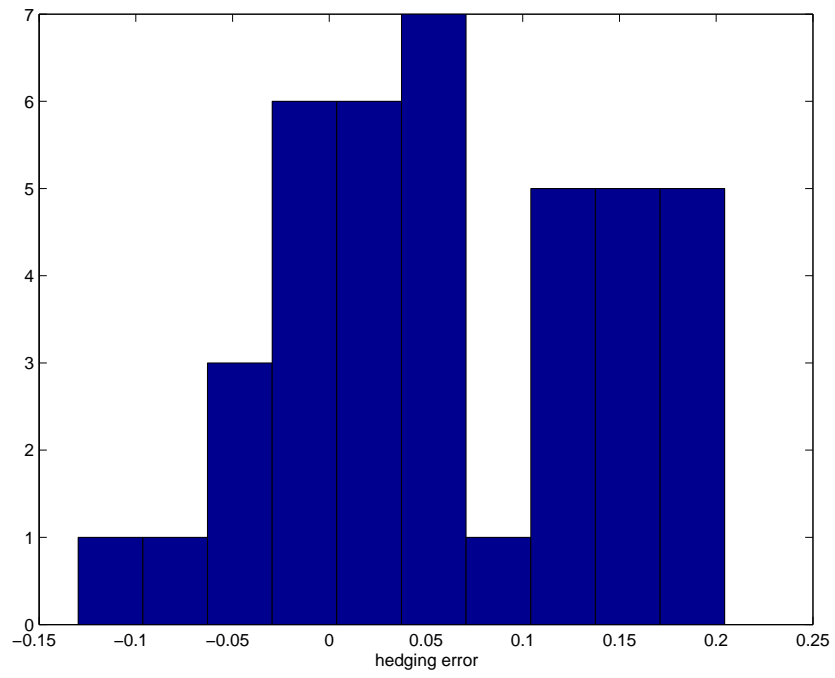


Figure 8: Histogram of hedging errors, arithmetic variance, 3m5y

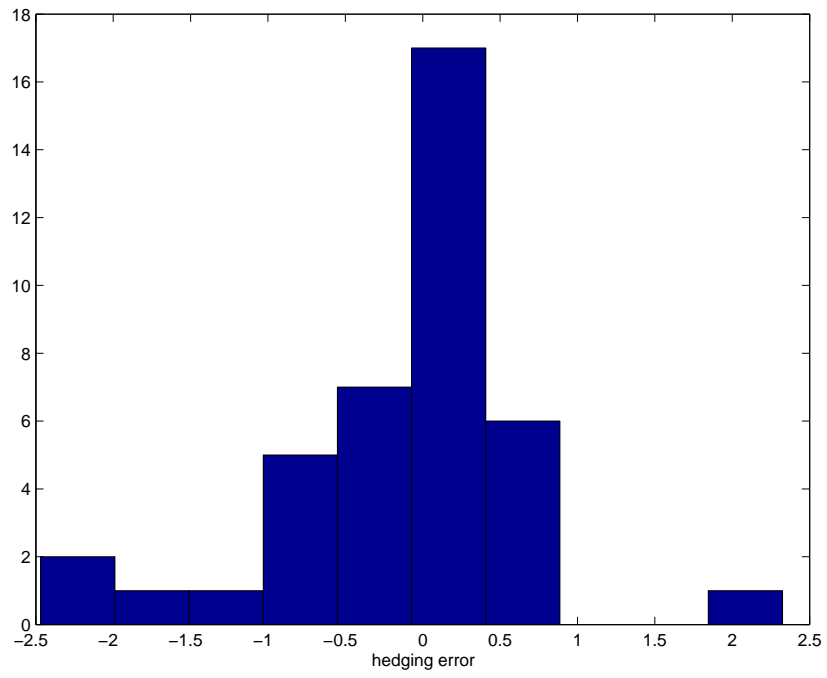


Figure 9: Histogram of hedging errors, arithmetic variance with quadratic hedge, 3m5y

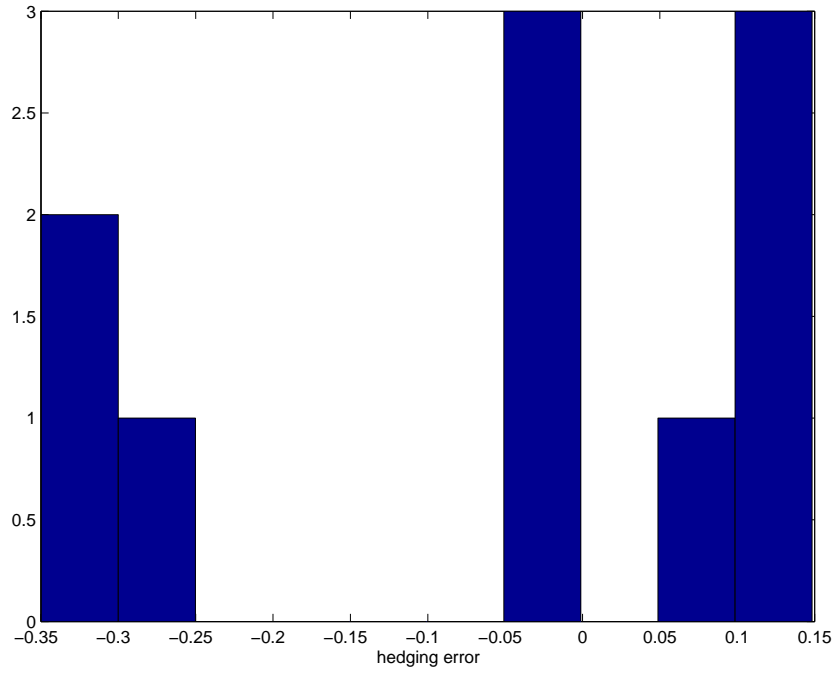


Figure 10: Histogram of hedging errors, arithmetic variance, 1y10y

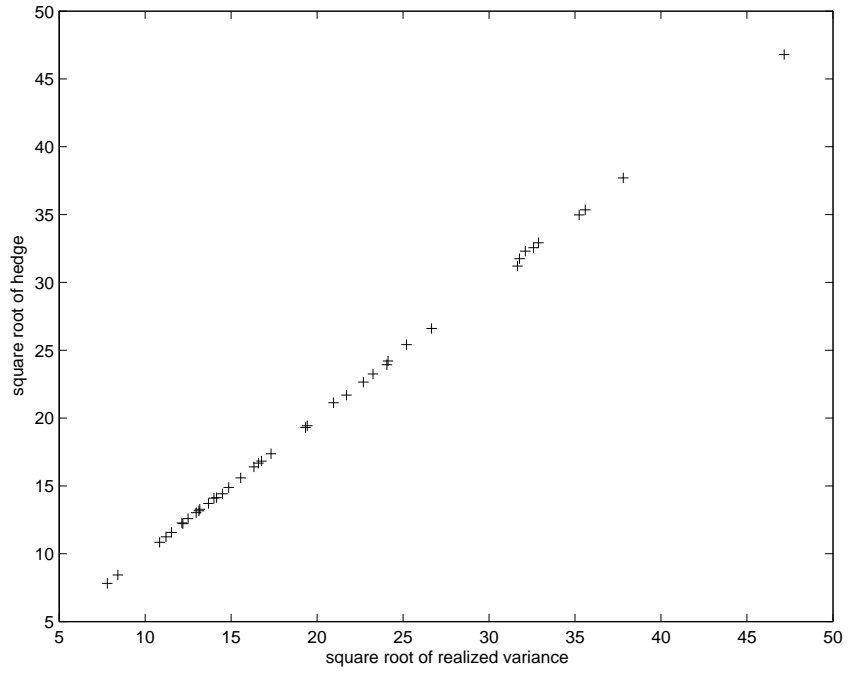


Figure 11: Realized geometric variance vs. hedge gain, 3m5y

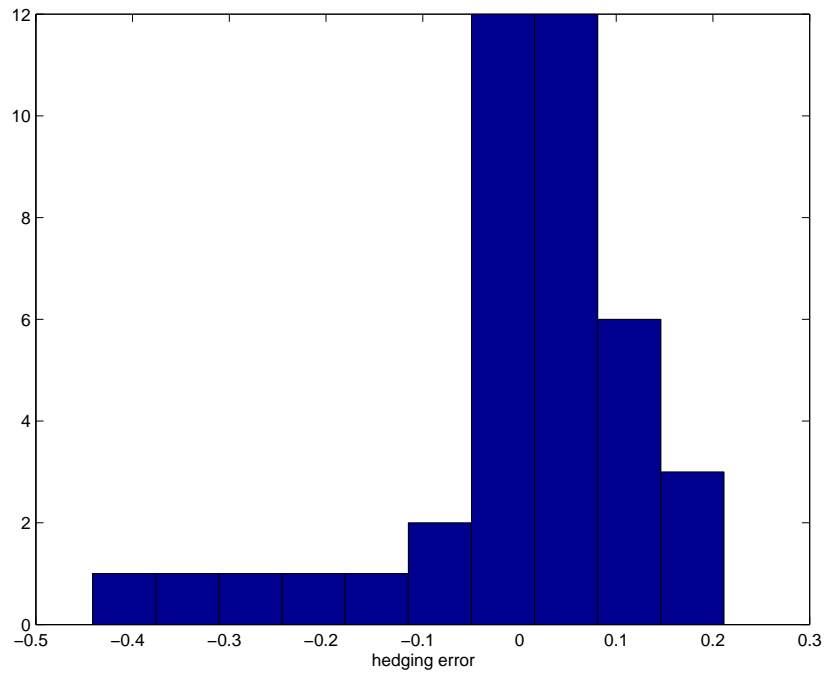


Figure 12: Histogram of hedging errors, geometric variance, 3m5y

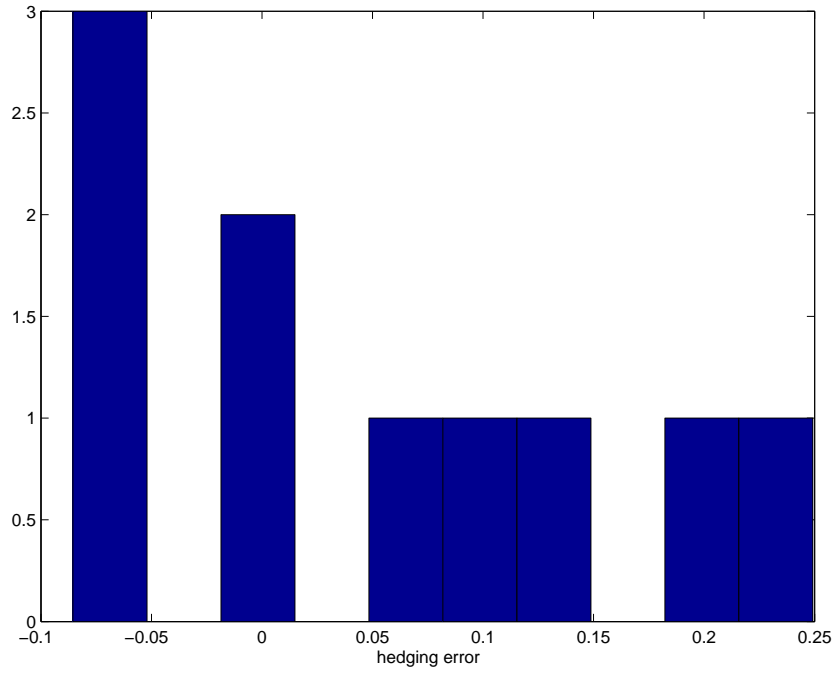


Figure 13: Histogram of hedging errors, geometric variance, 1y10y