Identification in Auction Models with Interdependent Costs*

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Abstract. I consider a single-unit procurement auction with asymmetric bidders, statistically dependent private information, and interdependent costs. When bidders are risk neutral, the model's payoff-relevant characteristics are: (i) the joint distribution of private information and (ii) each bidder's full information expected cost—the expected cost conditional on own and competitors' information. I show that these characteristics are nonparametrically identified from the distribution of bids conditional on observable cost shifters under the following four assumptions. First, each bidder's private information can be summarized by a real-valued signal. Second, the joint distribution of bidders' signals does not depend on the cost shifters. Third, each bidder's cost shifter affects his own full information expected cost but not his competitors'. Fourth, the observed data are generated by the repeated play of the same equilibrium where bidders use monotone pure strategies.

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1 Introduction

This paper provides an identification result for auction models with interdependent costs. Identification is achieved using variation in bidder-specific cost shifters. Cost shifters introduce variation in competitive conditions which is used to overcome the observational equivalence between models with private and interdependent costs, and to identify the payoff-relevant information structure of the model. Bidding behavior under varying competitive conditions is informative about the underlying information structure, as models with different structures predict different equilibrium responses to competition.

Procurement auction models can be classified in two broad categories depending on whether costs are private or interdependent. With interdependent costs, the information about each bidder's cost is scattered among all bidders. If a bidder learns the private information of one of his competitors', he can make a better forecast of his own cost. A rational bidder realizes that the results of the auction are informative about competitors' information. He wins the auctions that his competitors deem too costly or undesirable. His expected cost may be higher when the expectation is taken conditioning on the event where he wins. This adverse selection phenomenon is referred to as the winner's curse. With private costs, competitors' information do not help him to improve the forecast of any payoff-relevant characteristic of his cost distribution and the winner's curse is absent.

Winning in a more competitive environment means that the auction was considered undesirable by more bidders. With private costs, competition may reduce bidders' ability to profitably mark-up their bids but it does not change their expected cost conditional on winning; as a result, bidders are predicted to bid more aggressively.[1] Under interdependent costs and adverse selection, competition does increase their expected costs and they may not react to increased competition bidding as aggressively as with private costs. The effect of competitive conditions on bidding behavior is thus informative about the existence and magnitude of interdependent costs.

In an influential paper, Laffont and Vuong (1996) show that any joint distribution of bids that is rationalizable by an interdependent costs model is also rationalizable by some private costs model. This result holds for a fixed set of bidders and competitive conditions. Haile, Hong, and Shum (2004) develop a test for the null hypothesis of private costs against and alternative of interdependence using variation in competitive conditions: the number of bidders. In this paper, I provide a positive identification result using richer variation in competitive conditions: cost shifters.

I show that the payoff-relevant characteristics of the interdependent costs model are identified from the distribution of bids conditional on cost shifters. If bidders are risk neutral, these characteristics are the joint distribution of bidders private information and each bidder's full information

[1] The environment becomes more competitive if there are more bidders or if the existent bidders have lower costs. In the latter case, winning means that the auction was considered less desirable by the same set of bidders.

The result that more competition leads to more aggressive bidding in private cost auctions can fail due to strategic reasons (See Pinkse and Tan, 2005). The identification argument in this paper uses competitors' observed bidding behavior to separate strategic effects from the effect of costs interdependence.

cost: a function that returns the expected completion cost conditional on own and competitors' private information. These features of the economic environment are sufficient to analyze the effects of most policy changes (e.g., rules of the auction, reserve prices, subsidies) on outcomes such as bidding behavior, project allocation and prices. This information is not sufficient to analyze counterfactuals where the timing of the auction changes so that bidders are required to make decisions after some additional uncertainty in the model is resolved.

The identification result applies to auctions that satisfy the following three requirements. Bidders submit simultaneous bids. The project is awarded to whoever submits the lowest bid. The rule that determines the payment to each bidder as a function of all bids is continuous except where the project allocation changes, and satisfies some boundedness conditions. The first-price, second-price and all-pay sealed-bid auctions satisfy these requirements.

The result holds under four assumptions about the auction environment. First, each bidder's private information can be summarized by a real-valued random variable: a signal. Second, the joint distribution of bidders' signals does not depend on the cost shifters. Third, each bidder's cost shifter affects his own full information cost but not his competitors'. Fourth, the observed data are generated by the repeated play of the same equilibrium where bidders use monotone pure strategies. Under these conditions, I show that the payoff-relevant characteristics are identified over a subset of their domains that depends on observable features of the joint distribution of bids.

The identified set will be larger if the support of cost shifters is large, if cost shifters induce large variation in the joint distribution of bids, and if marginal cumulative distribution functions of bids are continuous and strictly increasing. Whether the data generated by repeated equilibrium play exhibit these features depends on the primitives of the model and the rules of the auction. I focus on the first-price sealed-bid auction with two bidders. Under some additional conditions on the primitives, the payoff relevant characteristics evaluated at any interior point of their domains are identified by the data generated by the repeated play of some equilibrium over a bounded set of cost shifters.

The rest of the paper is organized as follows. Section 2 describes the general interdependent cost framework. Section 3 illustrates the identification strategy with simple examples. Section 4 states the maintained assumptions along with the main theorems. Section 5 focuses on the first-price sealed-bid auction and derives conditions on the support of cost-shifters and the primitives of the model that ensure that the equilibrium distribution of bids satisfies the conditions for identification. Section 6 comments on the appropriateness of the assumptions for actual auction environments, and on how to use the results in this paper to test hypotheses and estimate the primitives of the model. The last section concludes.

2 The interdependent cost model

2.1 Primitives

An auctioneer procures the completion of a project and runs a sealed-bid auction between n risk-neutral bidders. The cost to bidder i is denoted by C_i, where upper-case letters are used to refer to random variables, and lower-case letters to refer to their realized values. His information is summarized by a signal S_i. At the time of the auction, i knows the realization of his own signal s_i, but is uncertain about the realization of the vector of competitors' signals $S_{-i} = \{S_j\}_{j \neq i}$ and own future project completion cost C_i. In other words, each bidder knows his own information but does not know his competitors'; moreover, his information only allows him to make an imperfect forecast of his own costs. Denote the full random vector of signals by $S = [S_i]_{i=1}^n$, and the vector of costs by $C = [C_i]_{i=1}^n$.

All bidders have access to the following public information: bidder-specific cost shifters $[x_1, x_2, ..., x_n]$, and a set of observable auction characteristics x_0. All public information is denoted by $x = [x_0, x_1, x_2, ..., x_n] \in X^o \subset \mathbb{R}^m$, $m \geq n+1$. X^o is the set of all possible vectors x. Cost shifters and auction characteristics are not necessarily one-dimensional. For example, the cost shifter of bidder i may include his distance to the project site and publicly observable predictors of his capacity constraints. Auction characteristics x_0 may include publicly available estimates of the cost and duration of the project. Observed heterogeneity x_0 can be conditioned upon and omitted from notation.

The primitives of the model are summarized by the joint distribution of costs and signals conditional on public information. Its cumulative distribution function (CDF) is denoted by: F (this is a shorthand notation for $F_{C,S|X}$). The distribution of bidder i's completion cost given his information at the time of the auction is denoted by $F_{C_i|s_i,x}$ and its expectation is $E(C_i|s_i, x)$. If he learns all competitors' signals his full information posterior distribution of costs becomes $F_{C_i|s_i,s_{-i},,x}$ with expectation $E(C_i|s_i, s_{-i}, x)$. This expectation is bidder i's full information expected completion cost or, for short, full information cost. All these posterior distributions and expectations of costs are uniquely determined by the primitive F.

The interdependent costs model nests both private and pure common costs paradigms. With private costs, each bidder knows his own expected completion cost, so i's full information cost does not depend on competitors' signals, i.e.,

$$E(C_i|s_i, s_{-i}, x) = E(C_i|s_i, x) \text{ for all } s_i, s_{-i} \text{ and } x. \tag{1}$$

With pure common costs, the cost of completing the project is common to all bidders, so they all share exactly the same full information cost. The model also allows for an nonindependent distribution of signals and asymmetric bidders.

Most auction models considered in the single-unit auction literature are special cases of the interdependent cost model where the joint distribution of costs and signals is restricted in some particular way. Write $F = F_{C|S,X}F_{S|X}$. Typical models impose conditions on either $F_{C|S,X}$ or $F_{S|X}$. In the traditional independent private values (costs) model $F_{C|S,X}$ satisfies the private values (costs) condition (1) and $F_{S|X}$ is the product of the marginal distributions $\left\{F_{S_i|X}\right\}_{i=1}^{n}$ (Milgrom and Weber, 1982, Section 2.1). The affiliated private values model allows signals to be affiliated (Li, Perrigne, and Vuong, 2002).[2] In the mineral rights model $C_i = C_j$ for all i and j, and signals have a particular correlation structure (Milgrom and Weber, 1982, Section 2.2).

2.2 Observables

Data consist of an independent sample of auctions. Signals and costs are drawn from F, which remains fixed across auctions. Each bidder submits a bid after observing his own private signal and all public information. The researcher observes these bids and all public information. Let B_i denote the bid made by i, $H_{B_i|X}$ its distribution conditional on public information, and H the joint distribution of bids conditional on public information (this is a shorthand notation for $H_{B|X}$). If bidder i decides not to participate in an auction, his bid is recorded as infinity; therefore, $\max_{b_i} H_{B_i|x}(b_i)$ may be less than one. X^o is the observed support of x, and X_i^o that of x_i. For simplicity, assume that $X^o = X_1^o \times ... \times X_n^o$.

Let $\mathcal{H}$ be the set of all conditional joint distributions of bids. H is a typical element of $\mathcal{H}$. When x is fixed, $H_{B|x}$ denotes the CDF of bids conditional on x implied by H. Similarly, $H_{B|x'}$ denotes the CDF of bids conditional on x' implied by the same H. It will be useful to define $H_{M_i|B_i=b_i,x}(t)$ as the CDF of $M_i = \min_{j\neq i} B_j$ conditional on $B_i = b_i, X = x$.

2.3 Identification

Following Roehrig (1988) and Athey and Haile (2002), a model is a set Ω of pairs $(F,\gamma) \in \mathcal{F} \times \Gamma$, where $\mathcal{F}$ is a set of joint distributions of costs and signals conditional on cost shifters, and Γ is a collection of mappings $\gamma : \mathcal{F} \to \mathcal{H}$. Each mapping γ returns a distribution of observables. Let Ch be a function that maps elements of $\mathcal{F} \times \Gamma$ to a space of characteristics. For example, bidder i's full information cost is a characteristic of F. In that case, $Ch(F) = E(C_i|S,X) = \int C_i dF_{C|S,X}$, where the integration is over the support of the joint distribution of costs.

Definition 1. *Characteristics $Ch(F,\gamma)$ are identified in Ω if for all $(F',\gamma') \in \Omega : \gamma'(F') = \gamma(F)$ implies $Ch(F',\gamma') = Ch(F,\gamma)$.*

$Ch(F,\gamma)$ is identified if the data unequivocally confirm it: there is no alternative explanation in Ω to the observed data where the primitive does not have characteristic $Ch(F,\gamma)$. If there is a transformation T such that $Ch(F,\gamma) = T \circ \gamma(F)$, then $Ch(F',\gamma') \neq Ch(F,\gamma)$ implies $\gamma'(F') \neq \gamma(F)$ which is equivalent to the definition of identification. Therefore, a sufficient condition for identification is that the characteristics can be expressed as a transformation of the observed data.

Definition 2. *Characteristics $Ch(F,\gamma)$ are testable in Ω if there is $H \in \mathcal{H}$ such that for all $(F',\gamma') \in \Omega : \gamma'(F') = H$ implies $Ch(F',\gamma') \neq Ch(F,\gamma)$.*

[2] Bidders' signals are affiliated if $f_S(s' \vee s) f_S(s' \wedge s) \geq f_S(s) f_S(s')$ for all s, s', where $\vee$ and $\wedge$ denote the component-wise maximum and minimum, respectively. Affiliation is a stronger notion of positive correlation. See Milgrom and Weber (1982) for more details.

$Ch\,(F, \gamma)$ is testable if it is possible that the data unequivocally reject it. There is a distribution of observables H that can not be explained by any $(F', \gamma') \in \Omega$ with characteristics $Ch\,(F, \gamma)$.

A well-known result in the empirical auctions literature is that the joint distribution of costs and signals $F_{C,S}$ is not identified from the joint distribution of bids H_B in the absence of additional observed variation (Laffont and Vuong, 1996). This paper provides a positive identification result. Define the payoff-relevant information structure of F as the following pair of characteristics:

$$\left\{ \{E\,(C_i|S, X)\}_{i=1}^{n}, F_{S|X} \right\}. \tag{2}$$

The rest of the paper is concerned with identification of this information structure.

A simple accounting of dimensions provide some intuition about the non identification result in Laffont and Vuong (1996) and the positive identification result in this paper. H_B is an n-dimensional joint distribution while the primitive of interest $F_{C,S}$ has dimension $2n$. In this paper, I close this gap by focusing on a coarser primitive and using additional n-dimensional variation in cost shifters.[3] The coarser primitive is (2). The observed data $H_{B|X}$ can be decomposed in the conditional copula of bids and n conditional marginal distributions, $\left\{H_{B_i|X}\right\}_{i=1}^{n}$. The conditional copula of bids identifies $F_{S|X}$; both objects are n-dimensional joint distributions conditional on n exogenous variables. After restricting each bidder's full information costs to depend only on his own cost shifter, full information costs have the same dimension as marginal distribution of bids conditional on X. Both $\left\{H_{B_i|X}\right\}_{i=1}^{n}$ and $\{E\,(C_i|S, X_i)\}_{i=1}^{n}$ are n functions of $n+1$ variables. However, this will not be sufficient for identification. It will be shown that identification of the full information costs requires that S is distributed independently from X. Because $F_{S|X}$ is identified from the conditional copula of bids, this is a testable restriction.

3 Examples

This section presents two simple examples that illustrate how the information structure is identified from the observed data. The assumptions here are invoked as needed; they are formally stated in Section 4.

Both examples use a condition for optimal response to competitors' strategies. The optimality condition states that the marginal expected revenue of choosing a bid that wins with a marginally larger probability should equal the marginal cost. The marginal revenue can be calculated from the distribution of competitors' bids. The marginal cost is the expected cost conditional on submitting a pivotal bid—a bid that ties with at least one competitor. In the two-bidder case presented in the first example, there is only one competitor to tie with; therefore, the event "tie with a competitor" has a simple representation in the space of competitor's signals. If competitor j uses a strictly monotone strategy, it is equivalent to "$S_j = s_j$". The identification argument is straightforward: each bid determines a unique marginal revenue that equals the full information cost evaluated at a particular set of signals. If $n > 2$, the event "tie with a competitor" has a more complex

[3] In fact, the dimensionality of the vector of cost shifters can be higher than n.

representation because there are more than one competitor to tie with; it takes the form "$S_j = s_j$ for some competitor j and $S_k \geq s_k$ for all other competitors". The second example describes how to recover the full information cost when there are three bidders. The procedure requires integrating and differentiating some measures of expected costs that can be recovered from the observed bidding behavior.

3.1 Two-bidder example

Identification of the information structure is easiest to illustrate in a second-price auction between two bidders: 1 and 2. Bidder 1 employs undominated strategies and submits a bid equal to his own expected cost conditional on own and public information and on the event where his bid is pivotal (see Milgrom and Weber, 1982; Athey and Haile, 2007). This event is the set of competitor's signals that prompts her to bid exactly the same amount as 1.

$$E\left(C_1|s_1, S_2 \in L, x\right) = b_1, \text{ where } L = \left\{s_2 : 2 \text{ bids } b_1 \text{ whenever he gets signal } s_2\right\}. \tag{3}$$

If L has only one element, the observed bid b_1 can be interpreted as the full information cost evaluated at a particular pair of signals. The first step in the identification argument is to show that the signal that prompted each observed bid can be recovered from observed bids. Identification of the joint distribution of bids follows immediately. The second step is to use competitor 2's cost shifter to identify 1's full information cost evaluated at different pair of signals. These steps are explained below.

First Step: recovering signals Assume that signals S_1 and S_2 are two uniformly-distributed, possibly correlated random variables.[4] Assume further that the observed data is the result of the repeated play of the same equilibrium where bidders employ strictly monotone pure strategies. The signal that makes bidder 1 bid b_1 is $s_1 = H_{B_1|x}\left(b_1\right)$. Similarly, the signal that makes bidder 2 bid b_1 is $s_2 = H_{B_1|x}\left(b_1\right)$. The statistical dependence between S_2 and S_1 is identified from dependence between B_2 and B_1 conditional on covariates x: $F_{S|x}\left(s\right) = P\left(S_1 \leq s_1, S_2 \leq s_2|x\right) = P\left(B_1 \leq b_1, B_2 \leq b_2|x\right)$.

The result above implies that (3) can be written as

$$E\left(C_1|H_{B_1|x}\left(b_1\right), H_{B_2|x}\left(b_1\right), x\right) = b_1. \tag{4}$$

The full information cost $E\left(C_1|s_1, s_2, x\right)$ equals b_1 such that $s_1 = H_{B_1|x}\left(b_1\right)$, and $s_2 = H_{B_2|x}\left(b_1\right)$.

Second Step: using variation in competitor's cost shifter Assume that x_2 is excluded from 1's full information cost. This exclusion restriction avoids confounding the effect of x_2 on the pairs of signals with a direct effect on costs. Equation (4) becomes

[4] The substantive assumption is that each signal is one-dimensional. The marginal distribution can be normalized to any distribution because signals have no cardinal meaning.

$$E\left(C_i|H_{B_1|[x_1,x_2]}\left(b_1\right), H_{B_2|[x_1,x_2]}\left(b_1\right), x_1\right) = b_1. \tag{5}$$

Evaluating this expression for different values of (x_2, b_1) results in the full information cost evaluated at different pairs of signals while x_1 is held constant. If there is a pair (x_2, b_1) such that $s_1 = H_{B_1|[x_1,x_2]}(b_1)$, and $s_2 = H_{B_2|[x_1,x_2]}(b_1)$, then the full information cost $E(C_1|s_1, s_2, x_1)$ is identified by b_1.

To obtain the full information for different s_2 keeping both x_1 and s_1 constant, choose x_2 so that

$$\Upsilon\left(x_2|s_1, x_1\right) = H_{B_2|[x_1,x_2]}\left(H^{-1}_{B_1|[x_1,x_2]}\left(s_1\right)\right) = s_2, \tag{6}$$

and choose $b_1 = H^{-1}_{B_1|[x_1,x_2]}(s_1)$; then $E(C_1|s_1, s_2, x_1)$ equals b_1. The mapping Υ returns the pivotal signal s_2 as a function of x_2. It depends only on the observed marginal distributions of bids. Variation in x_2 identifies 1's full information cost for different s_2 unless Υ is constant for all x_2.

The identification argument for first-price auctions is very similar. Campo, Perrigne, and Vuong (2003) show that the first-order optimality condition is

$$E\left(C_1|H_{B_1|[x_1,x_2]}\left(b_1\right), H_{B_2|[x_1,x_2]}\left(b_1\right), x_1\right) = b_1 - \frac{1 - H_{B_2|B_1,[x_1,x_2]}\left(b_1|b_1\right)}{h_{B_2|B_1,[x_1,x_2]}\left(b_1|b_1\right)}, \tag{7}$$

where $H_{B_2|B_1,[x_1,x_2]}(b_1|b_1)$ is the distribution function of B_2 conditional on $B_1 = b_1$, $X = [x_1, x_2]$ evaluated at b_1, and $h_{B_2|B_1,[x_1,x_2]}(b_1|b_1)$ is its density (which exists if strategies are strictly monotone and differentiable, and signals are uniformly distributed). The identification argument remains intact. If for the pair (s_1, s_2) there is a pair (x_2, b_1) such that $s_1 = H_{B_1|[x_1,x_2]}(b_1)$, and $s_2 = H_{B_2|[x_1,x_2]}(b_1)$, then the full information cost $E(C_1|s_1, s_2, x_1)$ is identified by the right hand side of (7): the bid b_1 minus the markup.

3.2 Three-bidder example

When there are more than two bidders it is also possible to recover the signal that prompted each bid and identify their joint distribution. The identification of the full information costs is complicated by the fact that optimality conditions do not impose a direct relationship between observables and full information costs. In second price auctions, for example, the optimality condition becomes

$$E\left(C_i|H_{B_i|[x_i,x_{-i}]}\left(b_i\right), \min_{j\neq i}\left(H^{-1}_{B_j|[x_i,x_{-i}]}\left(S_j\right)\right) = b_i, x_i\right) = b_i. \tag{8}$$

It is still true that i bids his own expected cost conditional on own and public information, and on the event where his bid is pivotal. However, this event is the set of competitor's signals such that at least one competitor bids b_i and all other competitors bid $b_j \geq b_i$. Consider a case with three bidders: 1, 2 and 3. The left-hand side of (8) denotes the expected cost over an L-shaped set of competitors' signals. Figure 1 shows the pair of competitors' signals that makes them both bid exactly b_1 along with an L-shaped set containing all competitors' signals such that their minimum bid is within $\varepsilon > 0$ from b_1. The left-hand side of (8) is the expected cost conditional on this set as $\varepsilon \to 0$ and can be written as:

$$\sum_{j=2,3} \left[\frac{h_{B_j|[x_1,x_{-1}]}(b) f(s_1,s_j) P(S_{-1j} \geq s_{-1j}|s_1,s_j)}{\sum_{k=2,3} h_{B_k|[x_1,x_{-1}]}(b) f(s_1,s_k) P(S_{-1k} \geq s_{-1k}|s_1,s_k)} \right] E(C_i|s_1,s_j,S_{-1j} \geq s_{-1j},x_1) \quad (9)$$

where $s_j = H_{B_j|x}(b_i)$ and $h_{B_j|x}$ is the density of $B_j|x$ for $j = 1,2,3$. The term in the square bracket is the probability that bidder i ties with bidder j conditional on tying with at least one competitor. These probabilities are identified from the observed distribution of bids. The objects of interest are the two terms $E(C_1|s_1,s_j,S_{-1j} \geq s_{-1j},x_1)$— the expected cost conditional tying with bidder j while underbidding the other bidder for $j = 2,3$. The optimality condition implies that each bid should equal a weighted average of these terms, which results in a single equation for two unknowns. While the information provided by a single bid is insufficient to identify these terms, it will be shown that variation in competitors' cost shifters can be used to identify the expected costs conditional on winning: $E(C_1|s_1,S_2 \geq s_2,S_3 \geq s_3,x_1)$.

Fix $\varepsilon > 0$, keep s_1 and x_1 constant, and use variation in $[x_2,x_3,b_1]$ to find a different L-shaped set that stacks on top of the previous one.[5] The expected cost conditional on the union of this two sets is equal to a weighted average of the expected cost conditional on each L-shaped set. The weights are given by the probability of each set. These probabilities are identified from the joint distribution of signals. This process can be repeated to obtain a weighted average over the whole rectangle $\{S_j \geq s_j\}_{j=2,3}$, as shown by Figure 2. If signals are independent from cost shifters, this average equals $E(C_1|s_1,S_2 \geq s_2,S_3 \geq s_3,x_1)$, which is the expected cost conditional on the event where 1 wins the auction. If $\varepsilon \to 0$, this average becomes an integration and the set of points in Figure 2 becomes a parametric curve over which the integration is performed. For a fixed ε, the position of the set points is determined by the thickness of each of the legs of the L-shaped sets. As $\varepsilon \to 0$, the slope of the curve at (s_2,s_3) converges to $\frac{h_{B_3|x}(b_1)}{h_{B2|x}(b_1)}$.

If $E(C_1|s_1,S_2 \geq s_2,S_3 \geq s_3,x_1)$ is identified around a neighborhood of (s_2,s_3), then its derivative is also identified. Differentiating it with respect to s_2 and s_3:

$$\frac{d^2 E(C_1|s_1,S_2 \geq s_2,S_3 \geq s_3,x_1) P(S_2 \geq s_2,S_3 \geq s_3|s_1)}{ds_2,ds_3} = E(C_1|s_1,s_2,s_3,x_1) f_{S_2,S_3|s_1}(s_2,s_3), \quad (10)$$

where $E(C_1|s_1,s_2,s_3,x_1)$ is the full information costs, and $f_{S_2,S_3|s_1}$ is the density of competitors' signals conditional on $S_1 = s_1$. The full information cost is obtained dividing the left hand side of (10) by the density of signals, both identified objects.

[5]The signal s_i is kept constant by choosing b_i so that $H_{B_i|[x_i,x_2,x_3]}(b_i) = s_i$ as x_2 and x_3 vary.

4 Identification

4.1 Auction Rules

The identification result applies to auctions where bidders' submit simultaneous or sealed bids, the project is awarded to whoever submits the lowest bid, and $p_i(b_i, b_{-i})$—the function that determines i's payment given own and competitors' bids—can be written as:

$$\begin{aligned} p_i(b_i, b_{-i}) &= p_i^0(b_i, b_{-i}) + p_i^1(b_i, b_{-i}) 1(i\ wins) \text{ if } b_i < \infty \\ &= p_i^\infty(b_{-i}) \text{ if } b_i = \infty \end{aligned} \tag{11}$$

where p_i^0, p_i^1 and p_i^∞ are continuous in all their arguments. When competitor j does not participate and bids $b_j = \infty$, $p_i^k(b_i, \{\infty, b_{-ij}\}) = \lim_{b_j \to \infty} p_i^h(b_i, b_{-i})$ for $k = 0, 1, \infty$. These functions are bounded with respect to competitors bids:

$$\sup_{b_{-i}} \left| p_i^k(b_i, b_{-i}) \right| < \infty, \text{ for } k = 0, 1, \infty. \tag{12}$$

For all b_i there exist a $K \geq 1$ such that for all $\left(b_i', b_{-i}', b_{-i}\right)$:

$$\frac{1}{K} \leq \frac{p_i^k(b_i', b_{-i}) - p_i^k(b_i, b_{-i})}{p_i^k\left(b_i', b_{-i}'\right) - p_i^k\left(b_i, b_{-i}'\right)} \leq K \text{ for } k = 0, 1. \tag{13}$$

This condition states that if the difference $p_i^k(b_i', b_{-i}) - p_i^k(b_i, b_{-i})$ is positive (negative) for some b_{-i}, it will be also positive (negative) and bounded away from zero for every b_{-i}'. If the difference is zero for some b_{-i}, it is zero for every b_{-i}. If $p_i^k(b_i, b_{-i})$ is additively separable in b_i and b_{-i}, (13) holds for $K = 1$.

Typical auction rules satisfy these conditions. First-price auctions have $p_i^0(b_i, b_{-i}) = 0$, $p_i^1(b_i, b_{-i}) = \min(b_i, r)$; second-price auctions have $p_i^0(b_i, b_{-i}) = 0$, $p_i^1(b_i, b_{-i}) = \min_{j \neq i}(b_j, r)$; and all-pay procurement auctions have $p_i^0(b_i, b_{-i}) = \min(b_i, 0)$, $p_i^1(b_i, b_{-i}) = r$.[6] In all cases, r is the maximum price the auctioneer is willing to pay.[7]

The main identification result does not depend on the tie-breaking rule because it is assumed that ties occur with probability zero in equilibrium. For the more general result, it will be assumed that ties are broken using a "priority rule": if $i < j$ and bidders i and j tie for the lowest bid then i is awarded the project. Therefore, i wins with b_i if $b_i < b_j$ for all $j < i$ and $b_i \leq b_j$ for all $j > i$.

4.2 Bayesian Game Representation

Sealed-bid auctions are modelled as Bayesian games. In the game theory jargon bidders are players, signals are types and bids are actions. The payoff functions and joint distribution of signals are

[6] Consider the random payment rule $\rho_i(b, w)$, where w is the realization of some random vector W. If W is independent of C, S, define $p_i(b) = E(\rho_i(b, W))$ and the results in this paper are unaltered.

[7] This framework accommodates bid preparation costs—a cost bidders pay after learning their signals. For example, in first-price sealed-bid auctions: $p_i^\infty(b_{-i}) = 0$, $p_i^0(b_i, b_{-i}) = -k_i$, and $p_i^1(b_i, b_{-i}) = \min(b_i, r)$. Bid preparation costs help to rationalize non-participation under some auction rules. This type of entry cost should be distinguished from an information-acquisition cost that is paid in order to learn the signal.

common knowledge. If bidder i is risk-neutral his payoff function is

$$u_i\left(b,s,x\right) = p_i\left(b\right) - E\left(C_i|s_i,s_{-i},x\right)1\left(i \text{ wins with } b_i\right), \tag{14}$$

where b_i is the bid submitted by bidder i, $b = [b_j]_{i=j}^n$.

A model primitive F implies a unique payoff-relevant information structure, which in turn implies a unique $\left\{\{u_j\}_{j=1}^n, F_{S|X}\right\}$. For each $x \in X^o$, these n payoff functions and the joint distribution of signals define a Bayesian game.[8]

4.3 Strategies and Equilibria

A monotone pure strategy for bidder i in the game defined by F and x is a nondecreasing real-valued function $\beta_i\left(s_i\right)$ that prescribes a bid for each s_i. This strategy varies across different games indexed by x. Thus, $\beta_i\left(s_i,x\right)$ describes how i behaves under different private and public information. A strategy profile $\{\beta_i\left(s_i,x\right)\}_{i=1}^n$ describes the behavior of all bidders in the family of games defined by F. Let $\mathcal{M}$ be a collection of such strategy profiles: $\beta \in \mathcal{M}$ if and only if $\beta_i\left(s_i,x\right)$ is nondecreasing in s_i for all i and x. Define the correspondence $m : \mathcal{F} \rightrightarrows \mathcal{M}$ such that $\beta = [\beta_i]_{i=1}^n \in m\left(F\right)$ if and only if $\{\beta_i\left(s_i,x\right)\}_{i=1}^n$ is a Bayes-Nash equilibrium in monotone pure strategies of the game defined by (F,x) for all x.[9]

It is said that β generates H if each marginal distribution of bids $H_{B_i|x}$ is generated by repeated play of the strategy profile β_i, i.e., if $H_{B_i|x}\left(b_i\right) = P\left(\beta\left(S_i,x\right) \leq b_i|x\right)$ for all $x \in X^o$, $i \in \{1,..,n\}$. Similarly, (β,F) generates H if the repeated play of strategy profile $\beta \in \mathcal{M}$ given the joint distribution of signals $F_{S|X}$ generates the joint distribution of bids $H \in \mathcal{H}$, i.e., if $H_{B|x}\left(b\right) = P\left(\cap_{i=1}^n \{\beta_i\left(S_i,x\right) \leq b_i\}|x\right)$ for all (x,b). Let $\psi : \mathcal{M} \times \mathcal{F} \to \mathcal{H}$ be such that $\psi\left(\beta,F\right) = H$ if and only if (β,F) generates H. Finally, let $\kappa : \mathcal{F} \rightrightarrows \mathcal{H}$ such that $\kappa\left(F\right)$ returns a set of conditional distributions of bids consistent with equilibrium play of monotone pure strategies,

$$\kappa\left(F\right) = \psi\left(m\left(F\right),F\right) = \{H \in \mathcal{H}|\exists\beta \in \mathcal{M} : H = \psi\left(\beta,F\right) \wedge \beta \in m\left(F\right)\}. \tag{15}$$

To summarize, m maps families of games to equilibrium strategy profiles in monotone pure strategies. ψ maps strategy profiles and joint distributions of signals to conditional joint distributions of bids $H \in \mathcal{H}$. κ, the composition of the previous two mappings, returns the conditional distributions of bids that are generated by the repeated play of some monotone pure strategy profile that constitutes an equilibrium of the game defined by (F,x) for all $x \in X^o$.

4.4 Assumptions

The true structure $(F,\gamma) \in \mathcal{F}\times\Gamma$ is assumed to have the following characteristics:

[8] Two different model primitives F and F' imply the same game for each $x \in X^o$ if and only if their payoff-relevant information structures are identical.

[9] If there are no equilibria in monotone pure strategies, this correspondence returns the empty set. Existence of such an equilibrium depends on the rules of the auction. I discuss the results on existence of equilibria in monotone pure strategies in Sections 5 and 6.

A.1. Signals are one-dimensional random variables. Without loss of generality, signals are marginally distributed uniform $[0,1]$. Thus,

$$F_{C,S|X} : \mathbb{R}_+^n \times [0,1]^n \times X^o \to [0,1] \,. \tag{16}$$

The joint distribution of signals, $F_{S|X}$, admits a continuous density function $f_{S|X}$. For every $x \in X^o$, there are two positive real numbers $\underline{f}$ and $\overline{f}$ such that $f_{S|x}(s) \in \left[\underline{f}, \overline{f}\right]$ for all $s \in [0,1]^n$.

A.2. Cost shifters and signals are independent: $F_{S|X} = F_S$.

A.3. The full information cost of bidder i is bounded, and continuous in (s, x_i). It is strictly increasing in x_i and s_i and does not depend on x_j for all $j \neq i$, i.e.,

$$E\left(C_i|s_i, s_{-i}, x\right) = E\left(C_i|s_i, s_{-i}, x_i\right). \tag{17}$$

A.4. The observed data result from the repeated play of the same equilibrium where all bidders employ monotone pure strategies: $H = \gamma(F) \in \kappa(F)$.

By assumptions A.1 and A.4 it is possible to recover the signal that prompted each observed bid; as a result, it is possible to condition on bidder i's own signal while varying competitors' cost shifters (see Section 3). Assumption A.3 imposes an exclusion restriction that avoids confounding the effect of competitors' cost shifters on competitive conditions with a direct effect on the full information cost. Conditional independence, $F_{C|S,X} = F_{C|S,X_i}$, is sufficient but not necessary for Assumption A.3. Assumption A.2 ensures that integrating over the space of competitors' signals results in the expected cost conditional on winning as discussed in Section 3.

Assumption A.4 implies that there exists an equilibrium in monotone pure strategies. This may signify different restrictions on F depending on the rules of the auction. In first-price auctions, for example, there exists an equilibrium in monotone pure strategies if signals are statistically affiliated. That is not true in second-price auctions (see Reny and Zamir, 2004). Assumption A.4 allows for the existence of multiple equilibria. It just requires that the data are generated by the repeated play of a single equilibrium where bidders employ monotone pure strategies.

The model under consideration is defined as the set of potential candidate structures having characteristics that are assumed, a priori, to apply to the true (F, γ):

Definition 3. *The model Ω_0 consists of all (F^*, γ^*) having characteristics A.1 to A.4. The model Ω_1 consists of all (F^*, γ^*) having characteristics A.1, A.3 and A.4. The model Ω_2 consists of all (F^*, γ^*) having characteristics A.1 and A.4.*

4.5 Results

Some of the results in the this section require that the observed distribution of bids satisfies some regularity conditions.

R.1. No probability mass at any bid: for all $x \in X^o$ and $i \in \{1,..,n\}$; $H_{B_i|x}$ is continuous on $\left[\underline{b}, \overline{b}\right]$, $H_{B_i|x}\left(\underline{b}\right) = 0$ and $H_{B_i|x}\left(\overline{b}\right) = 1$.

R.2. No probability mass at serious bids: for all $x \in X^o$, $i \in \{1,..,n\}$, and $t \in \mathbb{R}$; $\max_{j \neq i} H_{B_j|x}(t) < 1$ implies that $H_{B_i|x}$ is continuous at t.

R.3. Convex support of serious bids: for all $x \in X^o$, $i \in \{1,..,n\}$, and $t \in \mathbb{R}$; $\max_{j \neq i} H_{B_j|x}(t) < 1$ and $0 < H_{B_i|x}(t) < 1$ imply that $H_{B_i|x}$ is strictly increasing at t.

Condition R.1 states that there is no probability mass at any bid (including the bid ∞). This condition is useful to recover the signal that prompted each observed bid and identify $F_{S|X}$. Condition R.2 is weaker in that it requires no probability mass only at serious bids—those with a positive probability of winning. This condition suffices for the purposes of identifying the full information costs and F_S under the assumption that signals are independent of cost shifters. Condition R.3 ensures that for any β that generates H, $H_{B_i|x}(t) = s_i$ implies that $\beta_i(s_i, x) = t$ and that the first order condition for an s_i-type bidder evaluated at bid t identifies the full information cost conditional on the pivotal set.

Theorem 1 states that the joint distribution of signals is identified in Ω_2 if the observed joint distribution of bids conditional on cost shifters satisfies the no probability mass condition R.1. Theorem 2 states that the Assumption A.2 is testable in Ω_2. Theorem 3 shows that the full information cost is identified in Ω_0 if H satisfies R.2, R.3 and a large support condition: that cost shifters introduce enough variation in the distribution of bids. Consider the set of signals that generate an n-way tie. This set can be represented by a curve in $[0,1]^n$ (See Figure 3). Variation in competitors' cost shifters shifts this curve (See Figure 4). The large support condition requires that each point in a subset of $[0,1]^{n-1}$ belongs to a curve generated by some vector of competitors' cost shifters. Theorem 4 shows that the private costs hypothesis is testable in Ω_1.

If the observed distribution of bids does not satisfy the regularity conditions above, the joint distribution of signals and the full information costs will be identified for smaller subsets of their domains. Theorems 5 and 6 characterize the identified subsets without relying on regularity conditions. Sections 5 and 6 comment on whether different regularity conditions are shown to hold under some of the most studied auction rules.

Theorem 1. *If $\gamma(F)$ satisfies Condition R.1 then $F_{S|x}$ is identified in Ω_2 for all $x \in X^o$.*

Proof: Take any $x \in X^o$. Consider the characteristic $Ch(F) = F_{S|x}$ and let $H = \gamma(F)$. Take any $s = (s_1, ..., s_n) \in [0,1]^n$. Let b_i be such that $H_{B_i|x}(b_i) = s_i$ (b_i exists by Assumption A.1 and Condition R.1). Let $b = \{b_i\}_{i=1}^n$. Assumption A.4 implies that there is a $\beta \in m(F)$ that generates H. $\beta_i(s_i, x)$ is strictly increasing (otherwise $H_{B_i|x}(b_i)$ exhibits a discontinuity jump and violates Condition R.1). $\{S_i < s_i\}$ implies $\{\beta_i(S_i, x) \leq b_i\}$, which in turn implies $\{S_i \leq s_i\}$. Therefore,

$$\begin{aligned} F_{S|x}(s) &= P\left(\cap_{i=1}^n \{S_i \leq s_i\} | x\right) \\ &\geq P\left(\cap_{i=1}^n \{\beta_i(S_i, x) \leq b_i\} | x\right) = H_{B|x}(b) \\ &\geq P\left(\cap_{i=1}^n \{S_i < s_i\} | x\right) = F_{S|x}(s) \end{aligned} \tag{18}$$

The last equality follows from Assumption A.1. All the weak inequalities are equalities and $H_{B|x}(b) = F_{S|x}(s)$. There is a transformation of H that yields $F_{S|x}(s)$ for any $s \in [0,1]^n$ and any $x \in X^o$. $\square$

Theorem 2. *If there are x, $x' \in X^o$, Assumption A.2 is a testable characteristic of Ω_2.*

Proof: Let $Ch(F) = 1$ if F satisfies Assumption A.2. Take any $s = (s_1, ..., s_n) \in (0,1)^n$. Consider a $H \in \mathcal{H}$ that satisfies Condition R.2, and $H_{B|x}(b) \neq H_{B|x'}(b')$ where $b = [b_i]_{i=1}^n$, $H_{B_i|x}(b_i) = s_i$ for each i, $b' = [b'_i]_{i=1}^n$, and $H_{B_i|x'}(b'_i) = s_i$ for each i. Assumption A.1 and Condition R.2 and ensure that b and b' exist.

Consider any $(F^*, \gamma^*) \in \Omega_2$. Suppose $Ch(F^*, \gamma^*) = 1$ and $\gamma^*(F^*) = H$. Assumption A.4 implies that there is a $\beta \in m(F)$ that generates H. $\beta_i(s_i, x)$ is strictly increasing (otherwise $H_{B_i|x}(b_i)$ exhibits a discontinuity jump and violates Condition R.1). $\{S_i < s_i\}$ implies $\{\beta_i(S_i, x) \leq b_i\}$, which in turn implies $\{S_i \leq s_i\}$. It follows that $H_{B|x}(b) = F^*_{S|x}(s) = H_{B|x'}(b')$ which is a contradition. Thus, $Ch(F^*, \gamma^*) \neq 1$ or $\gamma^*(F^*) \neq H$. $\square$

Theorem 3. *Suppose that $\gamma(F)$ satisfies Conditions R.2 and R.3, $\sigma_{-i} \in [0,1)^{n-1}$, $s_i \in (0,1)$ and $x_i \in X_i^o$. If for all $s_{-i} \geq \sigma_{-i}$ there exist $(x_{-i}, t) \in X_{-i}^o \times \mathbb{R}$ such that $\left[H_{B_i|[x_i, x_{-i}]}(t)\right]_{i=1}^n = [s_i, s_{-i}]$, then $E(C_i|s_i, s_{-i}, x_i)$ is identified in Ω_0 for all $s_{-i} \geq \sigma_{-i}$.*

Corollary. *Suppose the conditions of Theorem 3. If $\left[H_{B_i|[x_i, x_{-i}]}(t)\right]_{i=1}^n$ as a function of (x_{-i}, t) is surjective (onto $[0,1]^n$), then $E(C_i|s, x_i)$ is identified in Ω_0 for all $s \in [0,1]^n$.*

Proof: Assumption A.3 implies that the full information cost is $E(C_i|s_i, s_{-i}, x_i)$ and Assumption A.2 implies that $f_{S_{-i}|s_i, x} = f_{S_{-i}|s_i}$. Let

$$\phi_i(s_{-i}|s_i, x_i) = E(C_i|s_i, S_{-i} \geq s_{-i}, x_i)\, P(S_{-i} \geq s_{-i}|s_i)\,. \tag{19}$$

The function ϕ_i returns the integral of

$$E(C_i|s_i, s_{-i}, x_i)\, f_{S_{-i}|s_i}(s_{-i}) \tag{20}$$

over the set $\{S_{-i} : S_{-i} \geq s_{-i}\}$. Assumption A.4 implies that there is a $\beta \in m(F)$ that generates H. Let

$$\begin{aligned} \overline{p}_i(b_i, s_i, x|\beta) &= E\left[p_i(b_i, \beta_{-i}(S_{-i}, x))\,|s_i, x\right], & (21)\\ \hat{p}_i\left(b_i, \tilde{b}_i, x|H\right) &= \int p_i(b_i, b_{-i})\, dH_{B_{-i}|B_i=\tilde{b}_i, x}(b_{-i}) & (22)\\ \eta_i(b_i, x|\beta) &= \left[P(\beta_j(S_j, x) \leq b_i)\right]_{j\neq i} \in [0,1]^{n-1} & (23) \end{aligned}$$

$\overline{p}_i$ is the revenue bidder i expects when he submits bid b_i, knows information (s_i, x) and believes that competitors behave according to β_{-i}. $\hat{p}_i$ is the expected revenue computed using the empirical distribution of bids $H_{B_{-i}|B_i=\tilde{b}_i, x}$. η_i is the vector of competitors' signals that generates an n-way tie at bid b_i when competitors bid according to β and the market conditions are x.

Define the nondecreasing curve $\lambda : [0,T] \to [0,1]^{n-1}$ as follows: (i) $\lambda(0) = s_{-i}$, (ii) for each $t \geq 0$ pick (b_t, x_t) such that $\left[H_{B_j|(x_i,x_t)}(b_t)\right]_{j=1}^{n} = [s_i, \lambda(t)]$ (the pair exists by hypothesis) (iii) $\lambda'(t)$ is a vector with typical element

$$\lambda_j'(t) = \lim_{k\to\infty} \frac{H_{B_j|x}\left(b^k\right) - H_{B_j|x}(b_i)}{H_{M_i|B_i=b_i,x}\left(b^k\right) - H_{M_i|B_i=b_i,x}(b_i)} \tag{24}$$

for some decreasing sequence $\left\{b^k\right\}_k$ such that $b^k \to b_t$.[10]

Let w_j^k be the probability that j submits a bid in $\left[b_i, b^k\right]$ conditional on there being at least one competitor who submits a bid in the same interval while all other competitors bid above b_i. Conditions R.2 and R.3 imply that ties occur with zero probability so that the conditional probability of two or more competitors bidding in the same interval goes to zero as $k \to \infty$. In the limit, the vector $\left[w_j^k\right]_{j\neq i}$ belongs to the $(n-1)$-dimensional simplex. Any decreasing sequence $\left\{b^k\right\}_k$ such that $b^k \to b_t$ has a subsequence such that $w_j^{k_q} \to w_j \in [0,1]$ for all j and $\sum_{j\neq i} w_j = 1$. It follows that:

$$\lambda_j'(t) = \frac{w_j}{f(s_i, s_j)\, P\left(S_{-ij} \geq s_{-ij} | S_i = s_i, S_j = s_j\right)}, \tag{25}$$

where $s_j = H_{B_j|x}(b_i)$.

The fact that $f(s_i, \lambda_j(t)) \leq \overline{f} < \infty$ implies that $\lambda_j'(t)$ is bounded away from zero and the curve λ eventually reaches $\lambda(T)$, such that $\lambda_j(T) = 1$ for some $j \neq i$. This curve can be constructed from the observed distribution functions and is identified from the data.

By the Fundamental Theorem of Integral Calculus:

$$\phi_i(\lambda(0)|s_i, x_i) = \phi_i(\lambda(T)|s_i, x_i) - \int_0^T \frac{d\phi_i(\lambda(t)|s_i, x_i)}{dt} dt. \tag{26}$$

Note that $\phi_i(\lambda(T)|s_i, x_i) = 0$, $\lambda(0) = s_{-i}$ and:

$$\begin{aligned} \frac{d\phi_i(\lambda(t)|s_i, x_i)}{dt} &= -\lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \int_{1(\lambda(t)\leq\tau\ngeq\lambda(t+\varepsilon))} E(C_i|s_i, \tau, x_i)\, f_{S_{-i}|s_i}(\tau)\, d\tau, \qquad (27)\\ &= -\lim_{k\to\infty} E\left(C_i | s_i, \eta_i(b_t, [x_i, x_t]|\beta) \leq S_{-i} \ngeq \eta_i\left(b^k, [x_i, x_t]|\beta\right), x_i\right) \\ &= -\lim_{k\to\infty} \frac{\overline{p}(b_t, s_i, [x_i, x_t]|\beta) - \overline{p}\left(b^k, s_i, [x_i, x_t]|\beta\right)}{P\left[\eta_i(b_t, [x_i, x_t]|\beta) \leq S_{-i} \ngeq \eta_i(b^k, [x_i, x_t]|\beta) | s_i\right]} \\ &= -\lim_{k\to\infty} \frac{\hat{p}(b_t, b_t, [x_i, x_t]|H) - \hat{p}\left(b^k, b_t, [x_i, x_t]|H\right)}{H_{M_i|B_i=b_t,[x_i,x_t]}\left(b^k\right) - H_{M_i|B_i=b_t,[x_i,x_t]}(b_t)} \end{aligned}$$

In the first equality, $1(\cdot)$ is an indicator function and $\lambda(t) \leq S_{-i} \ngeq \lambda(t+\varepsilon)$ is short-hand notation for $\lambda(t) \leq S_{-i}$ and not $S_{-i} \geq \lambda(t+\varepsilon)$—a subset of the space of competitors' signals that shrinks as $\varepsilon \to 0$. The second equality is implied by the definition of $\lambda'(t)$, the fact that $\lambda(t) = \left[H_{B_j|[x_i,x_t]}(b_t)\right]_{j\neq i} = \eta_i(b_t, [x_i, x_t]|\beta)$ and $P(S_{-i} \geq \eta_i(b_t, [x_i, x_t]|\beta)|s_i) = H_{M_i|B_i=b_t,[x_i,x_t]}(b_t)$. Lemma

[10]If the observed distribution of bids is differentiable this expression is simply $\frac{h_{B_j|x}(b_i)}{h_{M_i|B_i=b_i,x}(b_i)}$. The proof does not require differentiability.

2 in the Appendix shows that under Conditions R.2 and R.3 $\beta_i(s_i, [x_i, x_t])$ is strictly increasing and continuous at s_i. Thus, b_t is optimal for an s_i-type bidder when market conditions are $[x_i, x_t]$ and the third equality follows from Lemma 3 (part i). The optimality condition states that the increase in expected costs associated with a marginal increase in the probability of winning should be equal to the increase in the expected revenue. The fourth equality follows from Lemma 4. It states that the marginal revenue of an s_i-type bidder is identified by the empirical marginal revenue which can be constructed from the observed bidding behavior of his competitors conditional on his own bid b_t.

Using equations (26) and (27):

$$\phi_i(s_{-i}|s_i, x_i) = \int_0^T \lim_{k\to\infty} \frac{\hat{p}(b_t, b_t, (x_i, x_t)|H) - \hat{p}\left(b^k, b_t, (x_i, x_t)|H\right)}{H_{M_i|B_i=b_t,(x_i,x_t)}(b^k) - H_{M_i|B_i=b_t,(x_i,x_t)}(b_t)} dt, \tag{28}$$

where b_t and x_t are determined by the curve λ defined above. In first-price auctions, for example, the integrand in equation (28) is $b_t - \frac{1-H_{M_i|B_i=b_t,(x_i,x_t)}(b_t)}{h_{M_i|B_i=b_t,(x_i,x_t)}(b_t)}$. In second-price auctions it is just b_t.

Notice that the right hand side of (28) is a function of the observed data and known rules of the auction. Therefore, $\phi_i(\sigma_{-i}|s_i, x_i)$ is identified for all $\sigma_{-i} \geq s_{-i}$. Differentiating,

$$\frac{d^{n-1}\phi_i(s_{-i}|s_i, x_i)}{ds_{-i}} = E(C_i|s_i, s_{-i}, x_i) f_{S_{-i}|s_i}(s_{-i}). \tag{29}$$

The full information cost can be recovered dividing by the conditional density of signals which is identified.

If $\left[H_{B_i|[x_i,x_{-i}]}(t)\right]_{i=1}^n$ as a function of (x_{-i}, t) is onto $[0,1]^n$, the result above can be applied for any $s_i \in (0,1)$ and $\sigma_{-i} \in [0,1]^n$. By continuity of the full information costs, $E(C_i|s_i, s_{-i}, x_i)$ is also identified when $s_i \in \{0,1\}$ and the Corollary holds. □

Theorem 4. *If there are x, $x' \in X^o$ such that $x_i = x'_i$ and $x_{-i} \neq x'_{-i}$ then the private costs hypothesis is a testable characteristic of Ω_1.*

Proof: Let H be a conditional joint distribution of bids such that Conditions R.2 and R.3 are satisfied, and for some decreasing sequences $\left\{b^k\right\}_k$ and $\{b^q\}_q$ such that $b^k \to b_i$ and $b^q \to b'_i$:

$$\lim_{k\to\infty} \frac{\hat{p}(b_i, b_i, x|H) - \hat{p}\left(b^k, b_i, x|H\right)}{H_{M_i|B_i=b_i,x}(b^k) - H_{M_i|B_i=b_i,x}(b_i)} \neq \lim_{q\to\infty} \frac{\hat{p}(b'_i, b'_i, x'|H) - \hat{p}(b^q, b'_i, x'|H)}{H_{M_i|B_i=b'_i,x'}(b^q) - H_{M_i|B_i=b'_i,x'}(b_t)}, \tag{30}$$

where b_i and b'_i are such that $H_{M_i|B_i=b_i,x}(b_i), H_{M_i|B_i=b'_i,x'}(b'_i) < 1$, and $H_{B_i|x}(b_i) = H_{B_i|x'}(b'_i) = s_i$ for some $s_i \in (0,1)$. $\hat{p}$ is defined as in (22).

Let $Ch(F) = 1$ if F satisfies the private costs hypothesis. Consider any $(F^*, \gamma^*) \in \Omega_1$. Suppose $Ch(F^*) = 1$ and $\gamma^*(F^*) = H$. By Conditions R.2 and R.3, $H_{B_i|x}$ and $H_{B_i|x'}$ are continuous and strictly increasing at b_i and b'_i, respectively. By the same arguments as those used to show the last two equalities in (27) and imposing the private cost hypothesis, $E_{F^*}(C_i|s_i, x_i)$ should be equal to both sides of (30). This is a contradiction. Thus, $Ch(F^*) \neq 1$ or $\gamma^*(F^*) \neq H$. □

The next two theorems do not assume regularity conditions. $F_S(s)$ is shown to be identified in Ω_0 for all $s \in \mathcal{S}^o$, where $\mathcal{S}^o$ is a subset of its domain $[0,1]^n$. Similarly, bidder i's full information cost $E(C_i|s_i, s_{-i}, x_i)$ is shown to be identified in Ω_0 for all $(s, x_i) \in \mathcal{C}_i^o$, a subset of its domain $[0,1]^n \times X_i^o$. The sets $\mathcal{S}^o$, $\{\mathcal{C}_i^o\}_{i=1}^n$ are defined below. They depend on the support of cost shifters X^o and on the characteristics of the observed conditional distribution of bids H.

Let

$$\mathcal{S}^o = \left\{ s \in [0,1]^n \left| \exists x \in X^o, \forall i, \exists b_i : H_{B_i|x}(b_i) = s_i \right. \right\}. \tag{31}$$

There is no b_i such that $H_{B_i|x}(b_i) = s_i$ if $H_{B_i|x}$ exhibits a jump discontinuity or if it is below s_i for all b_i. In the first case, bidder i bids b_i with positive probability. In the second case, bidder i participates with probability less than s_i. A larger support of X^o implies a weakly larger set $\mathcal{S}^o$. If there is an $x \in X^o$ such that all bidders participate with certainty and employ strictly monotone strategies then $\mathcal{S}^o \in [0,1]^n$.

Theorem 5. *The joint distribution of signals $F_S(s)$ is identified in Ω_0 for all $s \in \mathcal{S}^o$.*

Proof: See Appendix.

Full information costs are identified if it is possible to integrate the expected cost conditional on L-shaped sets over the space of competitors' signals along a parametric curve (See Section 3 for two simple examples). If the conditions of Theorem 3 fail, it may still be possible to identify $E(C_i|s, x_i)$ for a subset of its domain. I introduce additional notation to deal with violations to the regularity conditions and characterize the set where the full information cost is identified. The main intuition is the same. Bidder i's optimality condition identifies the expected cost conditional on a set of competitors signals. Variation in x_{-i} introduces variation in these pivotal sets holding s_i and x_i fixed. Integrating over these pivotal sets identifies the expected costs conditional on winning.

Let Λ be the family of all nondecreasing and almost everywhere differentiable parametric curves in $[0,1]^{n-1}$. A curve is a function $[0,T] \to [0,1]^{n-1}$. Define the origin of a curve as $o(\lambda) = \lambda(0)$ and the destination as $d(\lambda) = \lambda(T)$. Define the concatenation operation: $\lambda_1, \lambda_2 \in \Lambda$, $(\lambda_1 + \lambda_2)(t) = \lambda_1(t)$ if $0 < t \leq T_1$, and $\lambda_2(t)$ if $T_1 < t \leq T_1 + T_2$. The resulting curve is continuous if λ_1 and λ_2 are continuous and if $d(\lambda_1) = o(\lambda_2)$. The remainder of this section characterizes the subset of $[0,1]^{n-1}$ where there is a continuous curve $\lambda \in \Lambda$ that can be used to integrate over the space of competitors signals. This curve may result from concatenating several curves λ_1,...,λ_K. Each of these curves defines a set of competitors signals, and the expected costs conditional on this set is identified from bidder i's optimality conditions.

If Condition R.3 fails and the support of serious bids is not convex, the observed $H_{B_i|x}$ will be constant for all b in an interval. The following two pseudo-inverses are defined to deal with that case:

$$\overline{H}_{B_i|x}^{-1}(s_i) = \inf\left\{b_i : H_{B_i|x}(b_i) > s_i\right\} \tag{32}$$

$$\underline{H}_{B_i|x}^{-1}(s_i) = \sup\left\{b_i : H_{B_i|x}(b_i) < s_i\right\} \tag{33}$$

$\overline{H}_{B_i|x}^{-1}(s_i)$ and $\underline{H}_{B_i|x}^{-1}(s_i)$ are the limiting optimal bids for bidder i as his signal approaches s_i from above and below, respectively (See Lemma 2 in the Appendix). By the continuity assumptions on bidders' full information costs, bidder i should be indifferent between these two limiting bids.

Let $\underline{b} = \underline{H}^{-1}_{B_i|x}(s_i)$ and $\bar{b} = \overline{H}^{-1}_{B_i|x}(s_i)$. Bidding $\underline{b}$ should result in a higher probability of winning, and higher expected costs. The indifference condition states that the increase in expected costs resulting from a lower bid should equal the increase in expected revenues. Revenues can be inferred from the data. The increase in expected costs is the expected costs conditional on the event where there is at least one competitor who bid between $\underline{b}$ and $\bar{b}$. This may be a positive probability event. Therefore, flat regions in the distribution of bids identify the expected cost conditional on a positive probability set of competitors signals. These sets have the form: $O \leq S_{-i} \ngeq D$; they are positive probability L-shaped sets. It is possible to define a curve λ with origin O and destination D, that corresponds to the set identified by this indifference condition. Define the following subsets of X^o:

$$\begin{aligned}
\overline{G}(s_i) &= \left\{x \in X^o : \lim_{t \to \bar{b}} H_{B_i|x}(t) = H_{B_i|x}\left(\bar{b}\right) \wedge \exists j \neq i : H_{B_j|x}\left(\bar{b}\right) > \underline{H}_{B_j|x}\left(\underline{b}\right)\right\}, \\
\underline{G}(s_i) &= \left\{x \in X^o : \lim_{t \to \underline{b}} H_{B_i|x}(t) = H_{B_i|x}\left(\underline{b}\right) \wedge \exists j \neq i : H_{B_j|x}\left(\bar{b}\right) > \underline{H}_{B_j|x}\left(\underline{b}\right)\right\},
\end{aligned}$$

where $\underline{H}_{B_j|x}(b_j) = P(B_j < b_j|x)$. Each element of $\overline{G} \cup \underline{G}$ contains information about the expected costs conditional on a set of the form: $O \leq S_{-i} \ngeq D$. The following notation is necessary to define the origin and destination associated with each of these sets:

$$\begin{aligned}
\overline{\Upsilon}(b_i|x,i) &= \left[H_{B_j|x}(b_i)\right]_{j \neq i}, \\
\underline{\Upsilon}(b_i|x,i) &= \left[\underline{H}_{B_j|x}(b_i)\right]_{j \neq i}.
\end{aligned} \tag{34}$$

Moreover, Υ^* reflects the priority rule: $\Upsilon^*_j = \overline{\Upsilon}_j$ if $j < i$ and $\Upsilon^*_j = \underline{\Upsilon}_j$ if $j > i$. The set of curves that correspond to these sets is given by:

$$G(s_i, x_i) = \left\{\begin{array}{c}
\lambda \in \Lambda | \exists x_{-i} \in X^o_{-i} : x = [x_i, x_{-i}] \in \overline{G}(s_i) \cup \underline{G}(s_i) \\
x \in \underline{G}(s_i) \implies o(\lambda) = \underline{\Upsilon}\left(\underline{b}|x,i\right) \\
x \notin \underline{G}(s_i) \implies o(\lambda) = \Upsilon^*\left(\underline{b}|x,i\right) \\
x \in \overline{G}(s_i) \implies d(\lambda) = \overline{\Upsilon}\left(\bar{b}|x,i\right) \\
x \notin \overline{G}(s_i) \implies d(\lambda) = \Upsilon^*\left(\bar{b}|x,i\right)
\end{array}\right\}$$

The origin (destination) vary depending on whether $H_{B_i|x}(t)$ is continuous at $\underline{b}$ $(\bar{b})$. If it is not continuous, bidding $\underline{b}$ $(\bar{b})$ may result in a tie with positive probability and it is necessary to follow the priority rule to determine the pivotal set. If it is continuous, the priority rule is irrelevant as ties at $\underline{b}$ $(\bar{b})$ occur with zero probability.

It is also possible to identify the expected costs conditional on a set of competitors signals of the form $O \leq S_{-i} \ngeq D$ by means of integrating zero probability sets (obtained from bidder i's first order condition) over a parametric curve. The subsets of X^o that convey information about zero probability sets are given by:

$$\begin{aligned}
\overline{Z}(s_i) &= \left\{x \in X^o : \lim_{t \to \bar{b}} H_{B_i|x}(t) = H_{B_i|x}\left(\bar{b}\right) \wedge \exists j \neq i : \forall t > \bar{b} \implies H_{B_j|x}(t) > H_{B_j|x}\left(\bar{b}\right)\right\} \\
\underline{Z}(s_i) &= \left\{x \in X^o : \lim_{t \to \underline{b}} H_{B_i|x}(t) = H_{B_i|x}\left(\underline{b}\right) \wedge \exists j \neq i : \forall t < \underline{b} \implies H_{B_j|x}(t) < H_{B_j|x}\left(\underline{b}\right)\right\}
\end{aligned}$$

Each element of $\overline{Z} \cup \underline{Z}$ identifies the expected costs over a zero probability set. The set of curves that can be constructed integrating over these zero probability sets is:

$$Z(s_i, x_i) = \left\{ \begin{array}{c} \lambda \in \Lambda | \ \forall t \in [0, T], \exists x_{-i} \in X_{-i}^o : x_t = [x_i, x_{-i}], \\ \text{either } x_t \in \underline{Z}(s_i), \lambda(t) = \underline{\Upsilon}(\underline{b}|x_t, i) \text{ and } \exists \{s_k\}_k \uparrow s_i \\ \lambda'(t) = \lim_{k \to \infty} \frac{\underline{\Upsilon}(\underline{b}|x_t, i) - \Upsilon^*\left(\overline{H}_{B_i|x}^{-1}(s_i^k)|x_t, i\right)}{\underline{H}_{M_i|B_i=\underline{b}, x_t}(\underline{b}) - H_{M_i|B_i=\overline{b}, x_t}^*\left(\overline{H}_{B_i|x_t}^{-1}(s_i^k)\right)}; \text{ or} \\ x_t \in \overline{Z}(s_i), \lambda(t) = \overline{\Upsilon}\left(\overline{b}|x_t, i\right) \text{ and } \exists \{s_k\}_k \downarrow s_i \\ \lambda'(t) = \lim_{k \to \infty} \frac{\Upsilon^*\left(\underline{H}_{B_i|x_t}^{-1}(s_i^k)|x_t, i\right) - \overline{\Upsilon}\left(\overline{b}|x_t, i\right)}{H_{M_i|B_i=\overline{b}, x_t}^*\left(\underline{H}_{B_i|x_t}^{-1}(s_i^k)\right) - H_{M_i|B_i=\overline{b}, x_t}\left(\overline{b}\right)}. \end{array} \right\}$$

The notation for $\underline{H}_{M_i|B_i=\underline{b},x}$ and $H_{M_i|B_i=\overline{b},x}^*$ follows the same conventions as the notation of $\underline{\Upsilon}$ and Υ^*.

The set that contains all concatenations of parametric curves that originate at s_{-i} and cross the set of competitors' signals until they reach a point in the boundary of $[0, 1]^{n-1}$ is:

$$L_i(s_{-i}|s_i, x_i) = \left\{ \begin{array}{c} \lambda \in \Lambda | \lambda = \sum_{k=1}^K \lambda_k, \ o(\lambda_1) = s_{-i}, \\ d(\lambda_K) \in \partial [0, 1]^{n-1}, \ d(\lambda_{k-1}) = o(\lambda_k) \text{ for all } k > 1 \\ \lambda_k \in G(s_i, x_i) \cup Z(s_i, x_i) \text{ for all } k = 1, ..., K \end{array} \right\} \tag{35}$$

The set of all (s, x_i) where there exist a parametric curve satisfying all these requirements is:

$$\mathcal{C}_i^o = \{(s, x_i) \in [0, 1]^n \times X_i^o : L_i(s_{-i}|s_i, x_i) \text{ is not empty}\}.$$

The interior of this set is:

$$int\mathcal{C}_i^o = \{(s_i, s_{-i}, x_i) \in \mathcal{C}_i^o | \exists \varepsilon > 0 : \|\sigma_{-i} - s_{-i}\| < \varepsilon \implies (s_i, \sigma_{-i}, x_i) \in \mathcal{C}_i^o \}.$$

The full information costs will be identified for all $(s, x_i) \in int\mathcal{C}_i^o$. The next theorem summarizes this result. The proof is relegated to the appendix.

Theorem 6. *The full information costs $E(C_i|s_i, s_{-i}, x_i)$ is identified in Ω_0 for all $(s, x_i) \in int\mathcal{C}_i^o$.*

5 First-price sealed-bid auctions

This section focuses on first-price sealed-bid auctions. It derives sufficient conditions on the primitives of the model and support of cost shifters for the existence of an equilibrium profile in monotone pure strategies that generates a conditional joint distribution of bids that satisfy the regularity conditions R.2 and R.3, and the support condition in Theorem 3. This distribution of bids identifies the full information costs and the joint distribution of signals of the model.

FPA.1. Signals are affiliated: $f_S(s' \vee s) f_S(s' \wedge s) \geq f_S(s) f_S(s')$ for all s, s', where $\vee$ and $\wedge$ denote the component-wise maximum and minimum, respectively.

FPA.2. Common Values: $E(C_i|s_i, s_{-i}, x_i)$ is strictly increasing in s_{-i}.

FPA.3. Full information costs are additively separable in cost-shifters and cost shifters are scalars: $E(C_i|S,x_i) = c_i(S) + x_i$ [11]

Reny and Zamir (2004) prove existence of an equilibrium in monotone pure strategies in interdependent costs (values) auctions under the assumption that signals are affiliated. Moreover, McAdams (2007) shows that if ties are broken using a "priority rule" all equilibria are monotone. Their result implies existence of a $\beta \in m(F)$.

Maskin and Riley (2000) prove that the support of the winning bid is a convex interval and that there are no atoms in the interior of its distribution. Their results are used to show that when $n = 2$ every $\beta \in m(F)$ features continuous bid functions and Condition R.3 holds for any H generated by β. I show that under the additional assumption of common values there exist a $\beta \in m(F)$ where bids functions are strictly increasing and Condition R.2 holds.

There are not many results in the theoretical literature on the effect of cost shifters on equilibrium bidding behavior. The available results apply to the independent private cost model. For example, Lebrun (1998) shows that if a bidder's cost distribution increases stochastically, so will his bid distribution. The requirement in Theorem 3 is stronger. There has to be a vector of cost shifters $x \in X^o$ for each vector of signals $s \in (0,1)^n$ such that, if s is realized under conditions x, there is a n-way tie. I construct a bounded set X^o that satisfies this requirement.

The following theorem summarizes the results of this section:

Theorem 7. *Consider a first-price sealed-bid auctions where ties are broken using a priority rule, there is no reserve price and bid preparation costs are zero. Assume that the primitives of the model satisfy Assumptions A.1 to A.3.*

i. If the primitives of the model also satisfy Assumptions FPA.1 and FPA.2, there exist a $\beta \in m(F)$ that generates a distribution H that satisfies Condition R.2; moreover, if $n = 2$, H also satisfies Condition R.3.

ii. Assume FPA.1 to FPA.3. For each $\sigma \in (0,1)^n$ and $x_i \in \mathbb{R}$, there exist a bounded set X^{σ}_{-i} such that for any $s_{-i} \geq \sigma_{-i}$ there are $x_{-i} \in X^{\sigma}_{-i}$, $t \in \mathbb{R}$, and $H \in \kappa(F)$ such that $\left[H_{B_i|[x_i,x_{-i}]}(t)\right]_{i=1}^{n} = [\sigma_i, s_{-i}]$.

Proof: See Appendix.

The support condition in Theorem 3 requires that there is a distribution of observables $H \in \kappa(F)$ such that for all $s_{-i} \geq \sigma_{-i}$ there exist a (x_{-i}, t) such that $\left[H_{B_i|[x_i,x_{-i}]}(t)\right]_{i=1}^{n} = [\sigma_i, s_{-i}]$. This is stronger than the conclusion of Part ii of Theorem 7 in that the distribution H has to be the same for all $s_{-i} \geq \sigma_{-i}$. Let $\left(H_{s_{-i}}, x_{s_{-i}}\right)$ be the bid distribution and cost shifter vector that satisfy the condition for s_{-i}. Theorem 7 imply that they exist. Let $\beta_{s_{-i}}$ be the strategy profile that generates $H_{s_{-i}}$. If $x_{s_{-i}} \neq x_{s'_{-i}}$ whenever $s_{-i} \neq s'_{-i}$ it is possible to construct $\hat{\beta}$ selecting different equilibrium strategies for different vectors of cost shifters such that: $\hat{\beta}\left(\cdot, x_{s_{-i}}\right) = \beta_{s_{-i}}\left(\cdot, x_{s_{-i}}\right)$. If it is not true

[11] More generally, $E(C_i|S,x_i) = c_i(S) + k_i(x_i)$. It is possible to renormalize cost shifters so that $x_i = k_i(x_i)$ after an appropriate re-normalization of cost-shifters)

that $s_{-i} \neq s'_{-i}$ implies $x_{s_{-i}} \neq x_{s'_{-i}}$, $\hat{\beta}$ can be constructed relying on an observable coordination variable q that distinguishes between different equilibria: $\hat{\beta}\left(\cdot, x_{s_{-i}}, q_{s_{-i}}\right) = \beta_{s_{-i}}\left(\cdot, x_{s_{-i}}\right)$. $q_{s_{-i}}$ should be observable to the econometrician so that she can identify the bid functions of each equilibrium played under the same vector of cost-shifters. If $\hat{H}$ is generated by $\hat{\beta}$, it satisfies support condition of Theorem 3, identifies $P\left(S_i \geq \sigma_i, S_{-i} \geq s_i\right)$ for any $s_{-i} \geq \sigma_{-i}$ (by Theorem 5), and identifies $E\left(C_i | \sigma_i, s_{-i}, x_i\right)$ for any $s_{-i} \geq \sigma_{-i}$ when $n = 2$ (by Theorem 3).

When $n > 2$ the result in Maskin and Riley (2000) does not imply condition R.3. It is possible that the support of the winning bid is a closed interval but that there are gaps in the support of bids of a particular bidder. One possible way to extend the results of this section to the $n > 2$ case is to find additional conditions on primitives that ensure that equilibrium bid functions do not exhibit jumps, and therefore that the support of each bidder's bids is a convex interval. Another possibility is to use Theorem 6 that takes advantage of jumps in the bid functions to achieve identification. If the equilibrium bid function exhibits a jump discontinuity from $\underline{b}$ to $\overline{b}$ at signal s_i, then an s_i-type bidder is indifferent between bidding $\underline{b}$ and $\overline{b}$. The proof of the theorem uses this indifference condition to perform the integration over the space of competitors signals and achieve identification. In order use Theorem 6 to show identification when $n > 2$ it is necessary to obtain a stronger result than Theorem 7.ii. The result should ensure that the bid level t that satisfies $\left[H_{B_i|[x_i, x_{-i}]}\left(t\right)\right]_{i=1}^{n} = \left[\sigma_i, s_{-i}\right]$ is equal to either $\underline{b}$ or $\overline{b}$.

6 Applying the identification result

The results in Theorems 1 to 6 apply to a particular auction market if its rules are consistent with the description in Section 4.1, Assumptions A.1 to A.3 are an appropriate characterization of the technological and informational environment, and Assumption A.4 describes how the observed data is generated. This section discusses the appropriateness of these Assumptions for actual auction environments and describes how to use the results in this paper to test hypotheses and estimate the primitives of the model.

6.1 Auction rules

Section 4.1 describes a game where only the winner pays the cost of completing the project, whoever submits the lowest bid wins, and the rule that determines the payment to each bidder as a function of all bids satisfies some regularity conditions. Standard first-price and second-price sealed-bid auctions fit this description.

Some procurement agencies award projects to preferred bidders even if they did not submit the lowest bid. In California, for example, state agencies apply a 5% discount to bids submitted by certified small businesses. The project is awarded to the bidder whose discounted bid is lowest, but the winner receives his undiscounted bid. These rules are consistent with the conditions in Section 4.1 after a simple redefinition of variables: b_i is the discounted bid and the payment function p_i includes a 5% premium over the bid when i wins.

While the notation and discussion in this paper suit a procurement auction, the results also apply in auctions where a good is sold. The expected full information cost is renamed expected full information utility (or value), the payoff function in (14) is multiplied by -1, and signals are appropriately redefined so that a higher signal implies a lower full information utility (or alternatively, if higher signals are meant to imply higher expected utility, the integration over the space of competitors' signals has to be performed towards the origin of the $(n-1)$-dimensional hypercube). Sealed-bid first-price and second-price auctions, as well as all-pay auctions, fit the description.

Auctions that award the project to the bidder that submits the second ranked bid, the median bid or the bid that is closest to the average do not fit the description.[12] Neither do auctions that allocate the project randomly among a subset of bidders (except when randomization is used as a tie-break rule). Multi-unit, combinatorial and dynamic auctions are beyond the scope of this paper.

6.2 Assumptions on the information structure

Identification requires that Assumptions A.1 to A.4 are satisfied by the true primitives of the market.

Assumption A.1 states that bidders summarize all private information in a one-dimensional variable. It rules out the possibility that bidder i receives two signals and that each signal affects the posterior distribution of competitors' information and costs in a different way. Suppose that each bidder receives two signals, one about his own equipment availability and one about conditions in the rental equipment market, and only the second signal changes the expectations about competitors' bidding behavior. This violates Assumption A.1.

Assumption A.2 does not allow the joint distribution of signals to depend on the vector of cost shifters. Assumption A.3 states that bidders' cost shifters only affect their own full information costs but not competitors'. For example, the distance of bidder j to the project does not affect the full information cost of bidder i. These assumptions are less innocuous than they might seem. They imply that j's cost shifter affects neither the technology of i nor the quality of the information of bidder j as perceived by i. They rule out the possibility that bidder i regards the signal of bidder j as more informative when j is more competitive. These assumptions may be reasonable in the context of highway procurement if costs uncertainties depend on input market conditions while cost shifters depend on observable bidder specific capacity constraints and distance to the project (Somaini, 2013). In the context of oil drainage leases, where the neighbor bidders are better informed than non-neighbors about the value of the tract, bidders' distances to the tract do not qualify as cost shifters that satisfy Assumptions A.2 and A.3 (Hendricks and Porter, 1988, 1993).

Assumption A.4 imply different restrictions on the primitives of the model depending on the rules of the auction. Reny and Zamir (2004) show that in the first-price sealed-bid auction if bidders have affiliated signals there exist an equilibria in monotone pure strategies (See Section 5). Affiliation rules out the existence of a common unobservable cost characteristic that affects

[12]See Conley and Decarolis (2013) for an empirical analysis of auctions with non-standard allocation rules

positively the cost of one bidder, but negatively that of another bidder. Reny and Zamir (2004) also show that in second-price auctions affiliation may cause Assumption A.4 to fail.

Throughout the paper it is assumed that bidders are risk neutral. This assumption can be justified in a particular setting on the basis that each project is small relative to the overall bidders' activity level.

6.3 Assumption on the data generating process

Assumption A.4 links bidding behavior observed in that data with that predicted by the model. It fails if the econometrician only observes a selected subset of auctions, if bidders play a different equilibrium in different plays of the same game, if the observed behavior does not correspond to any equilibrium, or if the equilibrium bidders play is not in monotone pure strategies.

Consider the case where a researcher observes a selected sample of auctions. If selection is based on the auction characteristics x_0 or the vector of cost shifters x, then X^o should be redifined to include only the vectors of cost shifters for which data is observed, and the assumption holds. If selection is based on the outcome of the auction which depends on the realization of signals (e.g. data includes only auctions with at least one finite bid) the observed H is not implied by equilibrium behavior, and the assumption fails.

Uniqueness of equilibria has only been proved for some particular cases (Lebrun, 1999; Bajari, 2001; Maskin and Riley, 2003). Suppose that the game defined by (F, x) admits q different equilibria and bidders play a different equilibrium each time. The observed $\gamma(F) = H$ will be a mixture over $\gamma^1(F)$, $\gamma^2(F)$,...,$\gamma^q(F)$, where γ^t is the mapping $\mathcal{F} \to \mathcal{H}$ implied by the t-th equilibrium. γ may not belong to $\kappa(F)$ even if $\gamma^t \in \kappa(F)$ for all $t \in 1, ..., q$. Assumption A.4 allows for the existence of multiple equilibria as long as for each vector of cost shifters the observed data is generated by the repeated play of the same equilibrium.

The observed bidding behavior may not correspond to any competitive Bayes-Nash equilibria if bidders are not rational, are colluding, or take into account features of the game that are omitted by the researcher.

Finally, even if there is an equilibrium in monotone pure strategies, bidders may be playing a nonmonotonic or mixed strategy equilibrium. McAdams (2007) shows that in first-price auctions under affiliation (and using the "priority rule" described in Section 4.1) all equilibria are monotone. These results do not apply to other auction rules.

6.4 Identified characteristics

The information structure of F contains all the information that is relevant at the time of the auction. It is enough to analyze the effects of most policy changes (e.g., rules of the auction, reserve prices, subsidies) on outcomes such as bidding behavior, project allocation and clearing prices. As long as the counterfactual situation does not imply that bidders make bidding or bargaining decisions after they learn additional information, their payoff functions still depend only on their full information costs and payment rule. Therefore, the equilibria of the counterfactual game can be calculated. For an example where bidders do make decisions after they learn additional information, consider the effects of allowing resale or subcontracting after bidders learn their costs. Suppose that at the subcontracting stage there is no private information and that each bidder's publicly known cost is his full information cost evaluated at the realized signals plus an idiosyncratic ex-post shock. The winner may make a take-it-or-leave-it offer to the competitor that has the lowest ex-post costs. His resale market opportunities are more profitable when the variance of the ex-post shock is larger because the expected minimum competitors' cost is lower. Because bidders should take into account the possibility of subcontracting (or resale as in Haile, 2001), this policy results in a more competitive auction environment if the variance of the ex-post shock is large. This variance is not an identified characteristic of the model. Therefore, it is not possible to compute this counterfactual using the identified information structure.

6.5 Testing and estimation

The constructive identification results in this paper suggest estimation procedures that do not require the computation of equilibrium strategies (similar to Guerre, Perrigne, and Vuong, 2000; Campo, Perrigne, and Vuong, 2003). Instead, the primitives of interest can be derived from a first-stage estimate of the joint distribution of bids conditional on cost shifters: $\hat{H}_{B|X}$. In the simplest case of two bidders and one-dimensional real cost shifters for each bidder, this is a function of four real variables. It is well-known that any nonparametric estimation of a four-dimensional distribution function is affected by the curse of dimensionality in typical sample sizes. While the estimation procedure could rely on parametric assumptions to smooth the data and overcome the curse, these assumptions are not necessary from an identification perspective.

Theorem 2 states that the assumption that the joint distribution of signals does not depend on cost shifters is testable. Under Assumptions A.1 and A.4 the joint distribution of signals equals the copula of bids conditional on a vector of cost shifters. Assumption A.2 implies that the copula of bids should not depend on the conditioning vector of cost shifters. Suppose that the copula is parametrized, and that it is possible to estimate its parameters and their asymptotic distribution for two different sub-samples selected on the basis of cost shifters. Under the null hypothesis that Assumption A.2 holds, the two sets of parameters should be equal.

If Assumption A.2 is not rejected by the data and is maintained, $H_{B|X}$ can be decomposed in the copula of bids—that is invariant with respect to cost shifters—and n marginal distribution of bids. The first step to estimate these objects is to estimate each bidders' marginal distribution of bids conditional on cost shifters: $\hat{H}_{B_i|X}$. The signal that prompts the observed bid b_i when cost shifters are x can be estimated by $\hat{s}_i = \hat{H}_{B_i|x}(b_i)$, which can be used to estimate the joint distribution of signals: $\hat{F}_S$ (see Theorems 3 and 6). $H_{B_i|X}$ and F_S have dimension $n+1$ and n, respectively. It may be necessary to assume some parametric functional form even in the simplest case with $n = 2$ and real cost shifters.

Theorem 4 states that the private cost hypothesis is testable. Under private costs, it is possible to recover the expected cost conditional on bidder's own information that rationalizes each observed bid.[13] The private cost hypothesis and Assumption A.3 imply that the distribution of the expected cost does not depend on competitors' cost shifters. The hypothesis can be tested obtaining the expected cost that rationalizes each observed bid under the null of private costs and testing whether the parameters that describe its distribution are invariant with respect to competitors' cost shifters. If the null hypothesis of private costs is rejected in the data, the results in this paper can be used to guide the estimation of the full information cost functions.

The information in $\left\{H_{B_i|x}\right\}_{i=1}^n$ suffices to derive the subset of the domain of full information costs that is identified. There is a simple graphical intuition for the two bidder case. Define the tie-curve for the vector of cost shifters x as the locus of points (s_1, s_2) in $[0,1]^2$ such that $s_1 = H_{B_1|x}(b)$ and $s_2 = H_{B_2|x}(b)$ for some $b \in \mathbb{R}$ (see Figure 3). Fix x_1, let x_2 vary and draw each resulting tie-curve (see Figure 4). Draw all curves such that $x' = (x_1, x_2')$ is an observed vector of cost shifters. Moreover, assume that $H_{B_1|x}(b)$ and $H_{B_2|x}(b)$ are sufficiently smooth in x so that it is possible to interpolate the tie-curve for all vectors x' within the support X^o and define a region in $[0,1]^2$ with all the pairs (s_1, s_2) that are part of a tie-curve for some $x' = [x_1, x_2'] \in X^o$ (see Figure 5). $E(C_1|s_1, s_2, x_1)$ is identified for every (s_1, s_2) in the region.

This intuition also applies to the $n > 2$ case. The tie-curve for x is a curve in $[0,1]^n$ that contains the points $[s_j]_{j=1}^n$ such that $s_j = H_{B_j|x}(b)$ for all j and some $b \in \mathbb{R}$. Fix x_i, let x_{-i} vary and draw each resulting tie-curve. Assume that the marginal distribution of bids conditional on cost shifters is smooth enough so that it is possible to interpolate the tie-curve for all vectors x' within the support X^o and define a region in $[0,1]^n$. $E(C_i|s_i, s_{-i}, x_i)$ is identified for all (s_i, s_{-i}) such that for all $\sigma_{-i} \geq s_{-i}$, (s_i, σ_{-i}) lies within the region (see Figure 6).

7 Conclusion

I provide an identification result for the payoff-relevant characteristics of the interdependent costs model. When bidders are risk neutral, these characteristics are the joint distribution of signals and the set of n full information costs. They are sufficient to analyze the effects of most policy changes (e.g., rules of the auction, reserve prices, subsidies) on outcomes such as bidding behavior, project allocation and prices. They are not sufficient to analyze counterfactuals where the timing of the auction changes so that bidders are required to make decisions after some additional uncertainty in the model is resolved.

The result applies to auctions that satisfy the following three conditions. Bidders' submit simultaneous bids. The project is awarded to whoever submits the lowest bid. The rule that determines the payment to each bidder as a function of all bids is continuouous except where the project allocation changes and satisfies some mild boundedness restrictions. The first-price, second-price and all-pay sealed-bid auctions satisfy these conditions. While the notation and discussion in this paper suit a procurement auction, the results also apply to auctions where a good is being

[13] In second-price auctions where bidders play undominated strategies, this cost equals the observed bid. In first-price auctions Campo, Perrigne, and Vuong (2003) show how to invert the bid function to recover the expected cost.

sold.

The identification result holds under the following assumptions. Each bidder's private information can be summarized by a real-valued signal. The joint distribution of bidders' signals is independent from cost shifters. Each bidder's cost shifter affects his own full information cost but not his competitors'. The observed data is generated by the repeated play of the same equilibrium where bidders use monotone pure strategies.

The payoff-relevant characteristics are identified over a subset of their domains that depends on the support of cost shifters and on the observed effect of cost shifters on bidding behavior. These subsets are larger when the support of the observed cost shifters is large and the marginal distribution of bids are continuous and strictly monotone. Some auction rules are more conducive to these conditions than others. I show that there exists a selection of equilibria in the two-bidder first-price sealed-bid auction that generates a conditional bid distribution that allows for identification provided that the support of cost shifters is sufficiently large. I also show that the support of cost shifters required for identification is always bounded.

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APPENDIX

A Proof of Theorem 5

The proof invokes the following Lemma:

Lemma 1. *If $H_{B_i|x}(b_i) = s_i$, S_i is uniformly distributed, and $\beta \in \mathcal{M}$ generates H: $\sigma_i > s_i$ implies $\beta_i(\sigma_i, x) > b_i$ and $\sigma_i < s_i$ implies $\beta_i(\sigma_i, x) \leq b_i$.*

Proof: Suppose $\beta_i(\sigma_i, x) \leq b_i$. Then, $\sigma_i \leq H_{B_i|x}(\beta_i(\sigma_i, x)) \leq H_{B_i|x}(b_i)$. The first inequality follows from monotonicity of β and the distribution of S_i, the second from monotonicity of the CDF. By the contrapositive, if $H_{B_i|x}(b_i) = s_i$ then $\sigma_i > s_i$ implies $\beta_i(\sigma_i, x) > b_i$.

Suppose $\beta_i(\sigma_i, x) > b_i$. Then, $H_{B_i|x}(b_i) = P(\beta_i(S_i, x) \leq b_i|x) \leq \sigma_i$. The equality follows from the definition of the CDF and the inequality from monotonicity of β and the distribution of S_i. By the contrapositive, if $H_{B_i|x}(b_i) = s_i$ then $\sigma_i < s_i$ implies $\beta_i(\sigma_i, x) \leq b_i$. □

Proof of Theorem 5: Take any $(s_1, ..., s_n) = s \in \mathcal{S}^o$. There are $x \in X^o$ and $b = \{b_i\}_{i=1}^n$ such that $H_{B_i|x}(b_i) = s_i$ for each bidder i. For all $\beta \in \mathcal{M}$ that generate H: $P(\beta_i(S_i, x) \leq b_i|x) = s_i$ and $H_{B|x}(b) = P(\cap_{i=1}^n \{\beta_i(S_i, x) \leq b_i\}|x)$. By Lemma 1, $P(\cap_{i=1}^n \{S_i < s_i\}|x) \leq H_{B|x}(b) \leq P(\cap_{i=1}^n \{S_i \leq s_i\}|x)$. By Assumption A.1 these two bounds are identical and by Assumption A.2 $F_S(s) = H_{B|x}(b)$. There is a transformation of H that yields F_S for any $s \in \mathcal{S}^o$. □

B Proof of Theorem 6

Define ϕ_i as in (19), $\overline{p}_i$ as in (21), $\hat{p}_i$ as in (22), and η_i as in (23). It will be useful to define:

$$\eta_i^*(b_i, x|\beta) = \left\{ \begin{array}{c} \{P(\beta_j(S_j, x) \leq b_i)\}_{j<i} \\ \{P(\beta_j(S_j, x) < b_i)\}_{j>i} \end{array} \right\} \tag{36}$$

Bid b_i wins when $s_{-i} > \eta_i^*(b_i, x|\beta)$ and loses when $s_{-i} \ngeq \eta_i^*(b_i, x|\beta)$. This definition of η_i^* differs from that of η_i in (23) in that it takes into account the priority tie-break rule. If $\lambda : \mathbb{R} \to [0,1]^{n-1}$ is a nondecreasing curve over the space of competitors' signals, $\lambda(t) \leq S_{-i} \ngeq \lambda(t+\varepsilon)$ is short-hand notation for $\lambda(t) \leq S_{-i}$ and not $S_{-i} \geq \lambda(t+\varepsilon)$—a subset of the space of competitors' signals that shrinks as $\varepsilon \to 0$. Figure 7 illustrates the shape of these sets for the $n = 3$ case. This is a pivotal set.

The following lemmas are used in the proof of the theorem:

Lemma 2. *For every $\beta \in \mathcal{M}$ that generates H: (i) $H_{B_i|x}$ continuous at b_i and $H_{B_i|x}(b_i) = s_i$ imply that $\beta_i(\cdot, x)$ is strictly increasing at s_i; (ii) $\underline{H}_{B_i|x}^{-1}(s_i) = \lim_{\sigma \uparrow s_i} \beta_i(\sigma, x)$ (iii) $\overline{H}_{B_i|x}^{-1}(s_i) = \lim_{\sigma \downarrow s_i} \beta_i(\sigma, x)$.*

Proof: $\beta_i(\cdot, x)$ is nondecreasing. Suppose that $\beta(\sigma, x) = b_i$ for all $\sigma \in [\underline{s}_i, \overline{s}_i]$ and that $s_i \in [\underline{s}_i, \overline{s}_i]$. $\lim_{t \uparrow b_i} H_{B_i|x}(t) \leq \underline{s}_i$ and $\lim_{t \downarrow b_i} H_{B_i|x}(t) \geq \overline{s}_i$. Continuity implies that $\underline{s}_i = \overline{s}_i = s_i$ which proves (i).

Consider some $b < \lim_{\sigma \uparrow s_i} \beta_i(\sigma, x)$. $b < \beta_i(\sigma, x)$ for some $\sigma < s_i$, thus $H_{B_i|x}(b) < s_i$. It follows that $\underline{H}^{-1}_{B_i|x}(s_i) \geq \lim_{\sigma \uparrow s_i} \beta_i(\sigma, x)$. Suppose that $\underline{H}^{-1}_{B_i|x}(s_i) > \lim_{\sigma \uparrow s_i} \beta_i(\sigma, x)$. There is a $b > \lim_{\sigma \uparrow s_i} \beta_i(\sigma, x)$ such that $H_{B_i|x}(b) < s_i$. There exist a $\sigma' \in \left(H_{B_i|x}(b), s_i\right)$ such that $\lim_{\sigma \uparrow s_i} \beta_i(\sigma, x) < b \leq \beta_i(\sigma', x)$, which contradicts monotonicity. This proves (ii).

Consider some $b > \lim_{\sigma \downarrow s_i} \beta_i(\sigma, x)$. $b > \beta_i(\sigma, x)$ for some $\sigma > s_i$, thus $H_{B_i|x}(b) > s_i$. It follows that $\overline{H}^{-1}_{B_i|x}(s_i) \leq \lim_{\sigma \downarrow s_i} \beta_i(\sigma, x)$. Suppose that $\overline{H}^{-1}_{B_i|x}(s_i) < \lim_{\sigma \downarrow s_i} \beta_i(\sigma, x)$. There is a $b < \lim_{\sigma \downarrow s_i} \beta_i(\sigma, x)$ such that $H_{B_i|x}(b) > s_i$. There exist a $\sigma' \in \left(s_i, H_{B_i|x}(b)\right)$ such that $\beta_i(\sigma', x) \leq b < \lim_{\sigma \downarrow s_i} \beta_i(\sigma, x)$, which contradicts monotonicity. This proves (iii). □

The following Lemma derives the implications of bidders playing best responses. If competitors' strategies are differentiable the optimality condition could be expressed as a first-order condition. Lemma 3 derives expressions that are similar to a first-order condition assuming that β_i is strictly increasing and continuous at s_i. By a revealed preference argument, $b_i^{-\mu} = \beta_i(s_i - \mu, x)$ results in a weakly higher expected payoff to an $(s_i - \mu)$-type bidder than $b_i^{+\mu} = \beta(s_i + \mu, x)$. The opposite holds for an $(s_i + \mu)$-type bidder. Letting $\mu \to 0$ results in a tight relationship between marginal expected costs and revenues.

Lemma 3. *Suppose that the joint distribution of signals and the full information costs are continuous and bounded with respect to s (as in Assumptions A.1 and A.3). Let $\beta_i(\cdot, x)$ be a best response to $\beta_{-i}(\cdot, x)$. Let $s_i \in (0, 1)$, $\underline{b} = \lim_{\sigma \uparrow s_i} \beta_i(\sigma, x)$ and $\overline{b} = \lim_{\sigma \downarrow s_i} \beta_i(\sigma, x)$. (i) If $\eta_i(t, x|\beta) > \eta_i(\overline{b}, x|\beta)$ for all $t > \overline{b}$, $\beta_i(\sigma, x) > \overline{b}$ for all $\sigma > s_i$, and $\left\{s^k\right\}_k$ is a strictly decreasing sequence such that $s^k \to s_i$ and $b^k = \beta\left(s_i^k, x\right)$, then*

$$\begin{aligned} &\lim_{q \to \infty} E\left(C_i | s_i, \eta_i\left(\overline{b}, x|\beta\right) \leq S_{-i} \not\geq \eta_i^*\left(b^{k_q}, x|\beta\right), x\right) \\ = \;&\lim_{q \to \infty} \frac{\lim_{\tau \downarrow \overline{b}_i} \overline{p}(\tau, s_i, x|\beta) - \overline{p}\left(b^{k_q}, s_i, x|\beta\right)}{P\left[\eta_i\left(\overline{b}, x|\beta\right) \leq S_{-i} \not\geq \eta_i^*\left(b^{k_q}, x|\beta\right)|s_i, x\right]}; \end{aligned} \tag{37}$$

for every subsequence $\left\{b^{k_q}\right\}_q$ such that this limit exists.[14] *(ii) If $\eta_i(t, x|\beta) < \eta_i(\underline{b}, x|\beta)$ for all $t < \underline{b}$, $\beta_i(\sigma, x) < \underline{b}$ for all $\sigma < s_i$, and $\left\{s^k\right\}_k$ is a strictly increasing sequence such that $s^k \to s_i$ and $b^k = \beta\left(s_i^k, x\right)$, then*

$$\begin{aligned} &\lim_{q \to \infty} E\left(C_i | s_i, \eta_i^*\left(b^{k_q}, x|\beta\right) \leq S_{-i} \not\geq \lim_{\tau \uparrow \underline{b}} \eta_i(\tau, x|\beta), x\right) \\ = \;&\lim_{q \to \infty} \frac{\overline{p}\left(b^{k_q}, s_i, x|\beta\right) - \lim_{\tau \uparrow \underline{b}} \overline{p}(\tau, s_i, x|\beta)}{P\left[\eta_i^*\left(b^{k_q}, x|\beta\right) \leq S_{-i} \not\geq \lim_{\tau \uparrow \underline{b}} \eta_i(\tau, x|\beta)|s_i, x\right]}; \end{aligned} \tag{38}$$

for every subsequence $\left\{b^{k_q}\right\}_q$ such that this limit exists. (iii) Let $b_i^{-\mu} = \beta_i(\sigma - \mu, x)$ and $b_i^{+\mu} =$

[14] The proof of Theorem 3 only requires part (i) of Lemma 3. Under the conditions of Theorem 3 η_i and $\overline{p}$ are continuous in b_i and $\eta_i^* = \eta_i$. Therefore, $\eta_i^*\left(b^{k_q}, x|\beta\right) = \eta_i\left(b^{k_q}, x|\beta\right)$, and $\lim_{\tau \downarrow \overline{b}_i} \overline{p}(\tau, s_i, x|\beta) = \overline{p}\left(\overline{b}_i, s_i, x|\beta\right)$.

$\beta_i(\sigma+\mu,x)$ for some $\mu>0$, if $\lim_{\mu\downarrow 0}\eta_i^*\left(b_i^{-\mu},x|\beta\right)\neq\lim_{\mu\downarrow 0}\eta_i^*\left(b_i^{+\mu},x|\beta\right)$:

$$E\left(C_i|s_i,\lim_{\mu\downarrow 0}\eta_i^*\left(b_i^{-\mu},x|\beta\right)\leq S_{-i}\ngeq\lim_{\mu\downarrow 0}\eta_i^*\left(b_i^{+\mu},x|\beta\right),x\right) \tag{39}$$
$$=\quad\frac{\lim_{\mu\downarrow 0}\overline{p}\left(b_i^{-\mu},s_i,x|\beta\right)-\lim_{\mu\downarrow 0}\overline{p}\left(b_i^{+\mu},s_i,x|\beta\right)}{P\left[\lim_{\mu\downarrow 0}\eta_i^*\left(b_i^{-\mu},x|\beta\right)\leq S_{-i}\ngeq\lim_{\mu\downarrow 0}\eta_i^*\left(b_i^{+\mu},x|\beta\right)|s_i,x\right]}.$$

and if $\lim_{\mu\downarrow 0}b_i^{-\mu}=\underline{b}<\infty$ but $b_i^{+\mu}=\infty$ for all $\mu>0$:

$$E\left(C_i|s_i,S_{-i}\geq\lim_{\mu\downarrow 0}\eta_i^*\left(b_i^{-\mu},x|\beta\right),x\right)=\lim_{\sigma\uparrow s_i}\frac{\lim_{\mu\downarrow 0}\overline{p}\left(b_i^{-\mu},s_i,x|\beta\right)-\overline{p}_i\left(\infty,x,s_i|\beta\right)}{P\left[S_{-i}\geq\lim_{\mu\downarrow 0}\eta_i^*\left(b_i^{-\mu},x|\beta\right)|s_i,x\right]}. \tag{40}$$

Proof: Consider the problem of s_i-type bidder when market conditions are x. His interim expected utility function is:

$$U_i\left(b_i,s_i,x|\beta\right)=\int u_i\left(\left\{b_i,\beta_{-i}\left(s_{-i},x\right)\right\},\left\{s_i,s_{-i}\right\},x\right)dF_{S_{-i}|s_i,x}\left(s_{-i}\right), \tag{41}$$

where $u_i(b,s,x)$ is defined in equation (14) and competitors' strategies $\beta_{-i}(\cdot,x)$ are taken as given. The interim expected utility function becomes:

$$U_i\left(b_i,s_i,x|\beta\right)=\overline{p}_i\left(b_i,x,s_i|\beta\right)-q_i\left(b_i,x,s_i|\beta\right), \tag{42}$$

where

$$q_i\left(b_i,x,s_i|\beta\right)\ =\ \int_{\eta_i^*(b_i,x|\beta)}^{[1,..,1]}E\left(C_i|s_i,s_{-i},x\right)f_{S_{-i}|s_i,x}\left(s_{-i}\right)ds_{-i}. \tag{43}$$

$$\begin{aligned}\overline{p}_i\left(b_i,x,s_i|\beta\right)\ &=\ \int_{[0,..,0]}^{[1,..,1]}p_i^0\left(b_i,\beta_{-i}\left(s_{-i}\right)\right)f_{S_{-i}|s_i,x}\left(s_{-i}\right)ds_{-i}]\\ &\quad+\int_{\eta_i^*(b_i,x|\beta)}^{[1,..,1]}p_i^1\left(b_i,\beta_{-i}\left(s_{-i}\right)\right)f_{S_{-i}|s_i,x}\left(s_{-i}\right)ds_{-i}\end{aligned} \tag{44}$$

$$\pi_i\left(b_i,x,s_i|\beta\right)\ =\ \int_{\eta_i^*(b_i,x|\beta)}^{[1,..,1]}f_{S_{-i}|s_i,x}\left(s_{-i}\right)ds_{-i}. \tag{45}$$

$\pi_i(b_i,x,s_i|\beta)$ is the probability that i wins with bid b_i given his information (x,s_i) and competitors' strategies β. Because x and β are fixed for the rest of the proof they are omitted from notation.

Consider any decreasing sequence $\left\{s^k\right\}_k$ such that $s^k\to s_i$, and construct $b^k=\beta\left(s_i^k,x\right)$. It follows that $b^k\downarrow\overline{b}$. By strict monotonicity of β_i, $b^k>\overline{b}$ for all k. Fix b^k. For all $q>k$ such that $\pi_i\left(b^q,s_i^q\right)>\pi_i\left(b^k,s_i^q\right)$, optimality of b^q when signal is s^q implies:

$$\frac{q_i\left(b^k,s_i^q\right)-q_i\left(b^q,s_i^q\right)+\overline{p}\left(b^q,s_i^q\right)-\overline{p}\left(b^k,s_i^q\right)}{\pi_i\left(b^q,s_i^q\right)-\pi_i\left(b^k,s_i^q\right)}\geq 0 \tag{46}$$

Consider the limit as $q\to\infty$. Boundedness of full information costs, payment functions and joint density of signal imply that the limit and integration operators commute (By the Dominated Convergence Theorem). By continuity of these functions with respect to i's own signal, the expression above becomes:

$$\frac{q_i\left(b^k,s_i\right)-\lim_{\tau\downarrow\bar{b}}q_i\left(\tau,s_i\right)+\lim_{\tau\downarrow\bar{b}}\overline{p}\left(\tau,s_i\right)-\overline{p}\left(b^k,s_i\right)}{\lim_{\tau\downarrow\bar{b}}\pi_i\left(\tau,s_i\right)-\pi_i\left(b^k,s_i\right)}\geq 0 \tag{47}$$

Define Ψ as the limit inferior of (47) as $k\to\infty$. $\Psi\geq 0$ by construction. Suppose that $\Psi>0$. The boundedness condition (13) ensure that the limit inferior of (47) is continuous with respect to s_i. There exist a $\delta>0$ such that for all $0<s_i'-s_i<\delta$

$$\liminf_{k\to\infty}\frac{q_i\left(b^k,s_i'\right)-\lim_{\tau\downarrow\bar{b}}q_i\left(\tau,s_i'\right)+\lim_{\tau\downarrow\bar{b}}\overline{p}\left(\tau,s_i'\right)-\overline{p}_i\left(b^k,s_i'\right)}{\lim_{\tau\downarrow\bar{b}}\pi_i\left(\tau,s_i'\right)-\pi_i\left(b^k,s_i'\right)}>\frac{\Psi}{2}>0$$

For any subsequence $\left\{b^{k_q}\right\}_q$, there exist a Q high enough so that for all $q>Q$:

$$\frac{q_i\left(b^{k_q},s_i'\right)-\lim_{\tau\downarrow\bar{b}}q_i\left(\tau,s_i'\right)+\lim_{\tau\downarrow\bar{b}}\overline{p}\left(\tau,s_i'\right)-\overline{p}_i\left(b^{k_q},s_i'\right)}{\lim_{\tau\downarrow\bar{b}}\pi_i\left(\tau,s_i'\right)-\pi_i\left(b^{k_q},s_i'\right)}>\frac{\Psi}{4}>0$$

Thus, $\lim_{\tau\downarrow\bar{b}}U_i\left(\tau,s_i',x|\beta\right)>U_i\left(b^{k_q},s_i',x|\beta\right)$. There exist a Q' high enough so that $Q'\geq Q$ and $b^{k_q}<\beta\left(s_i+\delta\right)$ for all $q>Q'$ (by strict monotonicity of β_i and because $b^{k_q}\to\bar{b}$). For all $q>Q'$: $\beta\left(\sigma\right)<b^{k_q}$ for all $\sigma\leq s_i$, $\beta\left(\sigma\right)\neq b^{k_q}$ for all $\sigma\in\left(s_i,s_i+\delta\right)$ and $\beta\left(\sigma\right)>b^{k_q}$ for all $\sigma\geq s_i+\delta$. This contradicts that $b^{k_q}=\beta\left(s_i^{k_q}\right)$ for some $s_i^{k_q}$. It follows that for any subsequence of $\left\{b^k\right\}_k$ the limit inferior (47) equals 0. Therefore,

$$\lim_{k\to\infty}\frac{q_i\left(b^k,s_i\right)-\lim_{\tau\downarrow\bar{b}}q_i\left(\tau,s_i\right)+\lim_{\tau\downarrow\bar{b}}\overline{p}\left(\tau,s_i\right)-\overline{p}_i\left(b^k,s_i\right)}{\lim_{\tau\downarrow\bar{b}}\pi_i\left(\tau,s_i\right)-\pi_i\left(b^k,s_i\right)}=0$$

and

$$\lim_{k\to\infty}\frac{\lim_{\tau\downarrow\bar{b}}q_i\left(\tau,s_i\right)-q_i\left(b^{k_q},s_i\right)}{\lim_{\tau\downarrow\bar{b}}\pi_i\left(\tau,s_i\right)-\pi_i\left(b_i^{k_q},s_i\right)}=\lim_{k\to\infty}\frac{\lim_{\tau\downarrow\bar{b}}\overline{p}\left(\tau,s_i\right)-\overline{p}_i\left(b^{k_q},s_i\right)}{\lim_{\tau\downarrow\bar{b}}\pi_i\left(\tau,s_i\right)-\pi_i\left(b_i^{k_q},s_i\right)} \tag{48}$$

for every subsequence $\left\{b^{k_q}\right\}$ such that these limits exist. Equation (37) follows from (48). Equation (38) is obtained considering any increasing sequence $\left\{s^k\right\}_k$ such that $s^k\to s_i$, constructing $b^k=\beta\left(s_i^k,x\right)$ and following the same argument as above.

Consider the case where $\eta_i^*\left(\underline{b}_i,x|\beta\right)\neq\eta_i^*\left(\overline{b}_i,x|\beta\right)$. Let $\mu>0$ and define $b_i^{-\mu}=\beta\left(s_i-\mu,x\right)$ and $b_i^{+\mu}=\beta\left(s_i+\mu,x\right)$. It follows that as $\mu\to0$, $b_i^{-\mu}\to\underline{b}_i$ and $b_i^{+\mu}\to\overline{b}_i$. By a revealed preference argument:

$$\overline{p}\left(b_i^{-\mu},s_i-\mu\right)-q\left(b_i^{-\mu},s_i-\mu\right)\geq\overline{p}\left(b_i^{+\mu},s_i-\mu\right)-q\left(b_i^{+\mu},s_i-\mu\right),$$

and

$$\overline{p}\left(b_i^{-\mu},s_i+\mu\right)-q\left(b_i^{-\mu},s_i+\mu\right)\leq\overline{p}\left(b_i^{+\mu},s_i+\mu\right)-q\left(b_i^{+\mu},s_i+\mu\right).$$

Letting $\mu\to0$, by continuity of both left and right-hand side with respect to s_i:

$$\lim_{\mu\to0}q\left(b_i^{-\mu},s_i\right)-\lim_{\mu\to0}q\left(b_i^{+\mu},s_i\right)=\lim_{\mu\to0}\overline{p}\left(b_i^{-\mu},s_i\right)-\lim_{\mu\to0}\overline{p}\left(b_i^{+\mu},s_i\right).$$

This implies equation (39). Equation (40) follows when $\overline{b}_i=\infty$ and $U_i\left(\overline{b}_i,s_i,x|\beta\right)=\overline{p}\left(\infty,s_i,x|\beta\right)$.

□

Lemma 4. *If Assumption A.1 holds, for all $\beta \in \mathcal{M}$ that generate H:*

$$\overline{p}_i(b, s_i, x|\beta) = \int p(b, b_{-i})\, d\mathcal{G}(b_{-i}), \tag{49}$$

where,

$$\mathcal{G}(b_{-i}) = P\left(\cap_{j\neq i}\left\{S_j \leq H_{B_j|x}(b_j)\right\}|s_i, x\right). \tag{50}$$

If $b_i \in \left\{\underline{H}^{-1}_{B_i|x}(s_i), \overline{H}^{-1}_{B_i|x}(s_i)\right\}$ and $H_{B_i|x}$ is continuous at b_i,

$$\mathcal{G}(b_{-i}) = H_{B_{-i}|B_i,x}(b_{-i}|b_i). \tag{51}$$

Proof: Take any $\beta \in \mathcal{M}$ that generates H. Define the mapping $T : [0,1]^{n-1} \to \mathbb{R}^{n-1}$ where $T(s_{-i}) = \{\beta_j(s_j, x)\}_{j\neq i}$. By the Change of Variable Theorem:

$$\begin{aligned}\overline{p}(b, s_i, x|\beta) &= \int p(b, \beta_{-i}(s_{-i}, x))\, dF_{S_{-i}|s_i,x}(s_{-i}) &(52)\\ &= \int p(b, b_{-i})\, d\mathcal{G}(b_{-i}), &(53)\end{aligned}$$

where $\mathcal{G}(b_{-i}) = P(\cap_{j\neq i}\{\beta_j(S_j, x) \leq b_j\}|s_i, x)$.

$H_{B_j|x}(b_j) = P(\{\beta_j(S_j, x) \leq b_j\}|x)$ because β generates H. By monotonicity of β_j:

$$\left[0, H_{B_j|x}(b_j)\right) \subset \{s_j \in [0,1]\,|\beta_j(s_j, x) \leq b_j|x\} \subset \left[0, H_{B_j|x}(b_j)\right].$$

Therefore,

$$P\left(\cap_{j\neq i}\left\{S_j < H_{B_j|x}(b_j)\right\}|s_i, x\right) \leq \mathcal{G}(b_{-i}) \leq P\left(\cap_{j\neq i}\left\{S_j \leq H_{B_j|x}(b_j)\right\}|s_i, x\right).$$

By Assumption A.1, the two bounds are equal and equation (50) follows.

$H_{B_i|x}$ is continuous at b_i and for all $\nu > 0$ either $H_{B_i|x}(b_i - \nu) < s_i$ or $H_{B_i|x}(b_i - \nu) > s_i$. By monotonicity of the two pseudo-inverses and continuity of the joint distribution of signals, conditioning on the zero-probability event $S_i = s_i$ is equivalent to conditioning on the events $\left\{B_i = \underline{H}^{-1}_{B_i|x}(s_i)\right\}$ or $\left\{B_i = \overline{H}^{-1}_{B_i|x}(s_i)\right\}$.

$$\begin{aligned}\mathcal{G}(b_{-i}) &= P(\cap_{j\neq i}\{\beta_j(S_j, x) \leq b_j\}|S_i = s_i, x)\\ &= P(\cap_{j\neq i}\{\beta_j(S_j, x) \leq b_j\}|\beta_i(S_i, x) = b_i, x)\\ &= H_{B_{-i}|B_i,x}(b_{-i}|b_i).\end{aligned}$$

□

Proof of Theorem 6: Take any $(s, x_i) \in \text{int}\mathcal{C}^o_i$. Pick $\lambda \in L_i(s_{-i}|s_i, x_i)$ and β that generates H.

$$\begin{aligned}\phi(s_{-i}|s_i, x_i) &= \phi(\lambda(0)|s_i, x_i) &(54)\\ &= \sum_{k=1}^{K} \phi(o(\lambda_k)|s_i, x_i) - \phi(d(\lambda_k)|s_i, x_i)\end{aligned}$$

The first equality follows because $o(\lambda) = s_{-i}$ and the second by continuity of λ. Take any $k \in \{1,..,K\}$. Suppose that $\lambda_k \in G(s_i, x_i)$, and let $x_k \in X^o$ be the corresponding vector of cost shifters:

$$\begin{aligned} & \phi\left(o\left(\lambda_k\right)|s_i, x_i\right) - \phi\left(d\left(\lambda_k\right)|s_i, x_i\right) \\ = \; & E\left(C_i|s_i, d\left(\lambda_k\right) \leq S_{-i} \ngeq o\left(\lambda_k\right), x_i\right) P\left(d\left(\lambda_k\right) \leq S_{-i} \ngeq o\left(\lambda_k\right)|s_i\right). \\ = \; & \lim_{\mu\downarrow 0} \overline{p}\left(b_k^{-\mu}, s_i, x_k|\beta\right) - \lim_{\mu\downarrow 0} \overline{p}\left(b_k^{+\mu}, s_i, x_k|\beta\right) \end{aligned}$$

The first equality follows from the definition of ϕ. Let $b_k^{-\mu} = \beta_i(s_i - \mu, x_k)$ and $b_k^{+\mu} = \beta_i(s_i + \mu, x_k)$. If β generates H, $\lim_{\mu\downarrow 0} \eta_i^*\left(b_i^{-\mu}, x|\beta\right) = o(\lambda_k)$ and $\lim_{\mu\downarrow 0} \eta_i^*\left(b_i^{+\mu}, x|\beta\right) = d(\lambda_k)$. The second equality is obtained using Lemma 3. Let $\underline{b}_k = \underline{H}_{B_i|x_k}^{-1}(s_i)$ and $\overline{b}_k = \overline{H}_{B_i|x_k}^{-1}(s_i)$. If $H_{B_i|x_k}$ is not continuous at $\underline{b}_k$, $\beta_i(s_i - \mu, x_k) = \underline{b}_k$ for μ sufficiently close to 0. If $H_{B_i|x_k}$ is continuous at $\underline{b}_k$ $\beta_i(s_i - \mu, x_k) > \underline{b}_k$ for all $\mu > 0$. Therefore,

$$\begin{aligned} x_k & \in \underline{G}(s_i) \implies \lim_{\mu\downarrow 0} \overline{p}\left(b_k^{-\mu}, s_i, x_k|\beta\right) = \lim_{t\uparrow \underline{b}_k} \overline{p}(t, s_i, x_k|\beta) = \lim_{t\uparrow \underline{b}_k} \hat{p}(t, \underline{b}_k, x_k|H) \\ x_k & \notin \underline{G}(s_i) \implies \lim_{\mu\downarrow 0} \overline{p}\left(b_k^{-\mu}, s_i, x_k|\beta\right) = \overline{p}(\underline{b}, s_i, x_k|\beta) = \hat{p}\left(\underline{b}_k, \overline{b}_k, x_k|H\right). \end{aligned}$$

The last equality follows from Lemma 4. Notice that if $x_k \notin \underline{G}(s_i)$, the expected revenue $\hat{p}$ is calculated conditioning on $\overline{b}_k$ instead of $\underline{b}_k$ because the event $\lim_{\varepsilon\to 0}\{B_i \in (\underline{b}_k - \varepsilon, \underline{b}_k + \varepsilon)\} = \{S_i \in (s_i - \mu, s_i)\}$ for some $\mu > 0$, but $\lim_{\varepsilon\to 0}\left\{B_i \in \left(\overline{b}_k - \varepsilon, \overline{b}_k + \varepsilon\right)\right\} = \lim_{\mu\to 0}\{S_i \in (s_i, s_i + \mu)\}$. Similarly,

$$\begin{aligned} x_k & \in \overline{G}(s_i) \implies \lim_{\mu\downarrow 0} \overline{p}\left(b_k^{+\mu}, s_i, x_k|\beta\right) = \lim_{t\downarrow \overline{b}_k} \overline{p}(t, s_i, x_k|\beta) = \lim_{t\downarrow \overline{b}_k} \hat{p}\left(t, \overline{b}_k, x_k|H\right) \\ x_k & \notin \overline{G}(s_i) \implies \lim_{\mu\downarrow 0} \overline{p}\left(b_k^{+\mu}, s_i, x_k|\beta\right) = \overline{p}\left(\overline{b}_k, s_i, x_k|\beta\right) = \hat{p}\left(\overline{b}_k, \underline{b}_k, x_k|H\right). \end{aligned}$$

It follows that $\phi(o(\lambda_k)|s_i, x_i) - \phi(d(\lambda_k)|s_i, x_i)$ is identified for all $\lambda_k \in G(s_i, x_i)$.

Suppose that $\lambda_k \in Z(s_i, x_i)$:

$$\begin{aligned} & \phi\left(o\left(\lambda_k\right)|s_i, x_i\right) - \phi\left(d\left(\lambda_k\right)|s_i, x_i\right) \\ = \; & -\int_0^T \frac{d\phi_i\left(\lambda_k(t)|s_i, x_i\right)}{dt} dt \qquad (55) \\ = \; & \int_0^T \lim_{\varepsilon\to 0} \varepsilon^{-1} \int_{\lambda_k(t) \leq \tau \ngeq \lambda_k(t+\varepsilon)} E\left(C_i|s_i, S_{-i} = \tau, x_i\right) f_{S_{-i}|s_i,x}(\tau)\, d\tau dt \\ = \; & \int_0^T \lim_{k\to\infty} E\left(C_i|s_i, \eta_i\left(\overline{b}_t, x_t|\beta\right) \leq S_{-i} \ngeq \eta_i^*\left(b^k, x_t|\beta\right), x\right) dt \\ = \; & \int_0^T \lim_{k\to\infty} \frac{\lim_{\tau\downarrow \overline{b}_i} \overline{p}\left(\tau, s_i, x_t|\beta\right) - \overline{p}\left(b^k, s_i, x_t|\beta\right)}{P\left[\eta_i\left(\tau, x_t|\beta\right) \leq S_{-i} \ngeq \eta_i^*\left(b^k, x_t|\beta\right)|s_i\right]} dt \\ = \; & \int_0^T \lim_{k\to\infty} \frac{\lim_{\tau\downarrow \overline{b}_i} \hat{p}\left(\tau, \overline{b}_t, x_t|H\right) - \hat{p}\left(b^k, \overline{b}_t, x_t|H\right)}{H^*_{M_i|B_i=\overline{b}_t, x_t}\left(b^k\right) - H_{M_i|B_i=\overline{b}_t, x_t}\left(\overline{b}_t\right)} dt \end{aligned}$$

The first equality follows from the Fundamental Theorem of Integral Calculus and the second from the definition of $\phi_i(\lambda_k(t)|s_i, x_i)$. Let $x_t \in X^o$ be the vector of cost shifters associated with t. Let $\bar{b}_t = \overline{H}^{-1}_{B_i|x_t}(s_i)$ and $b^k = \underline{H}^{-1}_{B_i|x_t}(s_i^k)$, where $\{s_i^k\}_k$ is the sequence that defines $\lambda'(t)$. Note that $\eta_i(\bar{b}_t, x_t|\beta) = \overline{\Upsilon}_i(\bar{b}_t|x_t)$ and $\eta_i^*(b^k, x_t|\beta) = \Upsilon_i^*(b^k|x_t)$. Assume that $x_t \in \overline{Z}(s_i)$ and $\lambda_k(t) = \overline{\Upsilon}_i(\bar{b}_t|x_t)$, then $\lambda'(t)$ can be written as:

$$\lambda'(t) = \lim_{k\to\infty} \frac{\eta_i^*(b^k, x_t|\beta) - \eta_i(\bar{b}_t, x_t|\beta)}{P\left[\eta_i(\bar{b}_t, x_t|\beta) \leq S_{-i} \ngeq \eta_i^*(b^k, x_t|\beta)|s_i\right]}$$

and the third equality follows. The fourth equality follows from Lemma 3 and the fifth from Lemma 4. If $x_t \in \underline{Z}(s_i)$ and $\lambda_k(t) = \underline{\Upsilon}_i(\underline{b}_t|x_t)$,

$$\lambda'(t) = \lim_{k\to\infty} \frac{\lim_{\tau\uparrow\underline{b}_t} \eta_i(\tau, x_t|\beta) - \eta_i^*(b^k, x_t|\beta)}{P\left[\eta_i^*(b^k, x_t|\beta) \leq S_{-i} \ngeq \lim_{\tau\uparrow\underline{b}_t} \eta_i(\tau, x_t|\beta)|s_i\right]},$$

where $\underline{b}_t = \underline{H}_{B_i|x_t}(s_i)$ and $b^k = \overline{H}^{-1}_{B_i|x_t}(s_i^k)$. The third equality should be modified accordingly. After applying Lemma 3 and Lemma 4, the integrand in the fifth equality becomes:

$$\lim_{k\to\infty} \frac{\hat{p}(b^k, \underline{b}_t, x_t|H) - \lim_{\tau\uparrow\underline{b}_t} \hat{p}(\tau, \underline{b}_t, x_t|H)}{\underline{H}_{M_i|B_i=\underline{b}_t, x_t}(\underline{b}_t) - H^*_{M_i|B_i=\underline{b}_t, x_t}(b^k)}$$

It follows that $\phi(o(\lambda_k)|s_i, x_i) - \phi(d(\lambda_k)|s_i, x_i)$ is identified.

It follows that all the terms in equation (54) are identified. Therefore, $\phi(s_{-i}|s_i, x_i)$ is identified. Differentiating $\phi_i(s_{-i}|s_i, x_i)$ with respect to s_{-i}:

$$\frac{d^{n-1}\phi_i(s_{-i}|s_i, x_i)}{ds_{-i}} = E(C_i|s_i, s_{-i}, x_i) f_{S_{-i}|s_i}(s_{-i}). \tag{56}$$

The full information cost can be recovered dividing by the density of signals $f_{S_{-i}|s_i}(s_{-i})$ which is identified. □

C Proof of Theorem 7

Proof of Part i Reny and Zamir (2004) show that there exists an equilibrium in pure monotone strategies provided that signals are affiliated. They construct a sequence of auction games where bidders are restricted to select bids from a finite set and show that for each game there is an equilibrium in monotone pure strategies. As the grid of available bids becomes dense in the real line, there exists a subsequence of equilibrium bid functions that converges to a set of monotone bid functions defined over the real line. They also restrict bidders with signals higher than $1-\varepsilon$ to bid infinity—not to participate—and they allow $\varepsilon \to 0$ as the number of available grid of bids becomes dense. They show that the limiting bid functions are an equilibrium of the game where all types of bidders are allowed to bid any real number. For any equilibrium with a finite number of bids, all bids (except non-participation) have positive probability of winning. Therefore, bidders profits conditional on winning should be nonnegative. By continuity of the full information costs, this property also applies to all bids in the unrestricted game.

Proposition 3 in Maskin and Riley (2000) ensures that the support of the distribution of the winning bid is an interval $[b_*, b^*]$. Proposition 4 in the same paper states that if there are two bidders that bid b^* with positive probability, at least one of them has to make zero profits.[15] Suppose that there are two bidders that bid b^* with positive probability and all bidders with higher priority stay out whenever they receive signals above $\overline{s}_k$. Formally, $\exists i, j$ such that $\beta(s_i) = \beta(s_j) = b^*$ for all $s_i \in [\underline{s}_i, \overline{s}_i]$ and $s_j \in [\underline{s}_j, \overline{s}_j]$, where $\underline{s}_i < \overline{s}_i$ and $\underline{s}_j < \overline{s}_j$. Assume that i has lower priority. For all $s_i \in [\underline{s}_i, \overline{s}_i]$ bidder i could discontinuously increase its probability of winning by reducing its bid by ε. His expected costs conditional on winning will be weakly lower because the set of competitors signals is now slightly better. The fact that this bidder chooses not to reduce his bid implies that $b^* \leq E(C_i|s_i, S_{-i} \geq \overline{s}_{-i}, x_i)$, for all $s_i \in [\underline{s}_i, \overline{s}_i]$. However, because strategies are the limit of a sequence of strategies where each bid has a positive probability of winning, $b^* = E(C_i|s_i, S_{-i} \geq \overline{s}_{-i}, x_i)$ for all $s_i \in [\underline{s}_i, \overline{s}_i]$, which contradicts strict monotonicity of s_i. Therefore, there is at most one bidder that bids b^* with positive probability: i bids b^* for all $s_i \in [\underline{s}_i, \overline{s}_i]$.

I will show that the existence of only one bidder that bids b^* leads to a contradiction. Suppose that bidder i would win with positive probability with a bid equal to $b^* + \varepsilon$ for an arbitrarily small $\varepsilon > 0$. If $\Pr(\min_{j \neq i} \beta_j(S_j) \in (b^*, b^* + \varepsilon)) = 0$, then bidder i has a profitable deviation: bid $b^* + \varepsilon$ instead of b^*. Therefore, $\Pr(\min_{j \neq i} \beta_j(S_j) \in (b^*, b^* + \varepsilon)) > 0$. This implies $\overline{s}_i = 1$, because there is a bidder j such that $\Pr(\beta_j(S_j) \in (b^*, b^* + \varepsilon)) > 0$ and $\overline{s}_i < 1$ imply that bidder j could win with a bid greater than b^* which contradicts b^* being the maximum of the support of the winning bid distribution. Let $b \in (b^*, b^* + \varepsilon)$ be such that $\beta_j(s_j) = b$ for some s_j. Because $b \geq E(C_j|s_j, S_{-j} \geq \eta(b|\beta))$, ε is arbitrary, i bids b^* with positive probability and $E(C_j|s_j, S_{-j} \geq s_{-j})$ is strictly increasing in s_{-j}: $b^* > E(C_j|s_j, S_{-j} \geq \eta(b^*|\beta))$ and j has a profitable deviation: bid slightly below b^*. Therefore, if i bids b^* for all $s_i \in [\underline{s}_i, \overline{s}_i]$ and $\varepsilon > 0$ bidder i has zero probability of winning with bid $b^* + \varepsilon$. It follows that all types $s_i \in [\underline{s}_i, \overline{s}_i]$ make zero profits when bidding b^*. Let $\{S_{-i} \geq \overline{s}_{-i}\}$ be the set of competitors' signals under which i wins with bid b^*. Because $b^* \geq E(C_i|\overline{s}_i, S_{-i} \geq \overline{s}_{-i}, x_i)$, $b^* > E(C_i|s_i, S_{-i} \geq \overline{s}_{-i}, x_i)$ for all $s_i \in [\underline{s}_i, \overline{s}_i)$. For any $\varepsilon > 0$, bidder i would have a positive probability of winning with bid $b^* - \varepsilon$, otherwise b^* would not be in the support of the distribution of winning bids. Therefore, for any $s_i \in [\underline{s}_i, \overline{s}_i)$ bidder i has a profitable deviation: bid slightly below b^*. It follows that there is no bidder that bids b^* with positive probability and that $H_{B_i|x}$ is continuous at every b in the support of the winning bids.

Let $n = 2$ and consider β that generates H such that for some t, $H_{B_j|x}(t) < 1$, $0 < H_{B_i|x}(t) = s_i < 1$ and $H_{B_i|x}(\tau) = s_i$ if and only if $\tau \in [\underline{b}, \overline{b}]$. Suppose that $\underline{b} < \overline{b}$. It follows that $t \in [\underline{b}, \overline{b}]$, β_i exhibits a jump discontinuity at $s_i \in (0,1)$: $\lim_{\sigma \uparrow s_i} \beta_i(\sigma) = \underline{b} < \overline{b} = \lim_{\sigma \downarrow s_i} \beta_i(\sigma)$. Consider bidder j's best response. No type bids in $(\underline{b}, \overline{b})$ because bidding $\overline{b}$ yields strictly higher payoff. Therefore, both bidders' bid functions have a jump discontinuity between $(\underline{b}, \overline{b})$. Let s_j be the signal at the discontinuity for bidder j. $H_{B_j|x}(t) < 1$ implies $s_j < 1$. Then $\underline{b} = b^*$ because $\underline{b}$ can be a winning bid and the support of the winning bid has to be an interval. However, type s_i can profitably deviate

[15] They analyze an auction where an object is sold and characterize the lowest endpoint, i.e. the bid with the least probability of winning that is actually submitted in equilibrium. In a procurement auction this is the highest endpoint.

bidding higher and winning with the same probability and in the same states of the world while earning higher revenues. Therefore, $\underline{b} = \overline{b}$ and $H_{B_i|x}$ is strictly increasing at t. $\square$

Proof of Part ii The proof of part ii of Theorem 7 uses results in Athey (2001) and Reny and Zamir (2004). The first step is to construct a discrete bid model and show that there exist a vector of cost shifters and an equilibrium in monotone pure strategies such that if H is generated by the repeated play of such equilibria $\left[H_{B_i|[x_i,x_{-i}]}(t)\right]_{i=1}^n = \{\sigma_i, s_{-i}\}$. This result follows from the Kakutani's fixed point theorem. The second step is to consider a sequence of fixed points as the grid of permissible bids becomes dense and show that it converges to a vector of cost shifters and a set of monotone strategies that constitute an equilibrium of the continuous bid auction and generate a joint distribution H such that $\left[H_{B_i|[x_i,x_{-i}]}(t)\right]_{i=1}^n = \{\sigma_i, s_{-i}\}$.

A discrete bid model. Let $\mathcal{A}_i = \{a_0 < a_1 < ... < a_M\}$ be the sets of available bids to bidder i. Let $[\underline{s}_i, \overline{s}_i] \subset [0,1]$ be a subset of i's signals. A monotone pure strategy $\beta_i : [\underline{s}_i, \overline{s}_i] \to \mathcal{A}_i$ can be represented a step function (See Athey, 2001) that describes the points in $[\underline{s}_i, \overline{s}_i]$ at which β_i jumps. The behavior of i at the jump points is inconsequential. Let $\Sigma_i = \left\{t \in [\underline{s}_i, \overline{s}_i]^M \,| \underline{s}_i \leq t_1 \leq t_2 \leq ... \leq t_M \leq \overline{s}_i\right\}$; $t \in \Sigma_i$ represents β_i if $t_m = \inf\{\sigma|\beta_i(\sigma) \geq a_m\}$.

When competitors are restricted to select bids from a discrete set and employ monotone strategies $\{\beta_j\}_{j\neq i}$, their strategies can be represented by $T_{-i} \in \Sigma_{-i} = \prod_{j\neq i}\Sigma_j$, where $T_{-i} = \{t_j\}_{j\neq i}$ and $t_j \in \Sigma_j$. Similarly, $T \in \Sigma = \prod_{i=1}^n \Sigma_i$ represents the strategies of all bidders. The event where i wins with bid b given competitors strategies represented by T_{-i} will be denoted by: $\eta_i(b|T_{-i})$. The utility of bidder i when competitors use strategies represented by T_{-i} will be denoted by $U_i(b_i, s_i, x_i|T_{-i})$. This notation stresses that competitors are bidding from a discrete set of bids. Define bidder i's best response correspondence when restricted to choose from the set of bids $\mathcal{A}$ as:

$$b_i^*(s_i, x_i, T_{-i}, \mathcal{A}) = \arg\max_{b\in\mathcal{A}} U_i(b, s_i, x_i|T_{-i}).$$

Define the subset of Σ_i that represents monotone best response $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ as:

$$T_i^{BR}(x_i, T_{-i}, \mathcal{A}, [\underline{s}_i, \overline{s}_i]) = \{t \in \Sigma_i : \forall s_i \in [\underline{s}_i, \overline{s}_i], t_m < s_i < t_{m+1} \implies a_m \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})\}.$$

The following results are used in the proof:

Lemma 5. *If $E(C_i|s, x_i)$ is nondecreasing in x_i then $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ is nondecreasing in the strong set order in x_i.*

Proof: Consider $b' > b$ and $x_i' > x_i$. Let $\pi = P(S_{-i} \geq \eta_i(b|T_{-i})|s_i)$ denote the probability of the event where i wins with bid b given competitors strategies represented by T_{-i}. Define π'

analogously for bid b'

$$\begin{aligned}
& U_i\left(b', s_i, x_i'|T_{-i}\right) - U_i\left(b, s_i, x_i'|T_{-i}\right) \\
= \; & b'\pi' - b\pi + \int_{1(\eta_i(b|T_{-i}) \leq \tau \ngeq \eta_i(b'|T_{-i}))} E\left(C_i|s_i, S_i = \tau, x_i'\right) f_{S_{-i}|s_i}(\tau)\, d\tau \\
\geq \; & b'\pi' - b\pi + \int_{1(\eta_i(b|T_{-i}) \leq \tau \ngeq \eta_i(b'|T_{-i}))} E\left(C_i|s_i, S_i = \tau, x_i\right) f_{S_{-i}|s_i}(\tau)\, d\tau \\
\geq \; & U_i\left(b', s_i, x_i|T_{-i}\right) - U_i\left(b, s_i, x_i|T_{-i}\right)
\end{aligned}$$

The function $U_i(b, s_i, x_i|T_{-i})$ exhibits increasing differences in (b, x_i); therefore $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ is nondecreasing in the strong set order in x_i (by Topkis Theorem). $\square$

Lemma 6. *Because $U_i(b, s_i, x_i|T_{-i})$ is continuous in (s_i, x_i, T_{-i}), the graph of $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ as a function of (s_i, x_i, T_{-i})[16] is closed for any $\mathcal{A}$.*

Proof: Consider a sequence $\left(b^k, s_i^k, x_i^k, T_{-i}^k\right)$ that converges to (b, s_i, x_i, T_{-i}) such that $b^k \in b_i^*\left(s_i^k, x_i^k, T^k, \mathcal{A}\right)$. There is a K, such that for all $k > K$, $b^k = b$ and $U_i\left(b, s_i^k, x_i^k|T_{-i}^k\right) \geq U_i\left(a, s_i^k, x_i^k|T_{-i}^k\right)$ for all $a \in \mathcal{A}$. By continuity, it follows that $U_i(b, s_i, x_i|T_{-i}) \geq U_i(a, s_i, x_i|T_{-i})$; thus, $b \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$. $\square$

Lemma 7. *If the graph of $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ as a function of (s_i, x_i, T_{-i}) is closed for all $s_i \in [\underline{s}_i, \overline{s}_i]$, then the graph $T_i^{BR}(x_i, T_{-i}, \mathcal{A}, [\underline{s}_i', \overline{s}_i'])$ as a function of $(x_i, T_{-i}, \underline{s}_i', \overline{s}_i')$ is also closed for all $[\underline{s}_i', \overline{s}_i']$ strictly included in $[\underline{s}_i, \overline{s}_i]$.*

Proof: Consider a sequence $\left(\underline{s}_i^k, \overline{s}_i^k, x_i^k, T_{-i}^k, t^k\right)$ that converges to $(\underline{s}_i, \overline{s}_i, x_i, T_{-i}, t)$ such that $t^k \in T_i^{BR}\left(x_i^k|T_{-i}^k, \mathcal{A}, \left[\underline{s}_i^k, \overline{s}_i^k\right]\right)$ for all k. Consider signal $s_i \in [\underline{s}_i, \overline{s}_i]$ such that $t_m < s_i < t_{m+1}$ for some $m \in \{0, ..., M\}$. Because t_m^k and t_{m+1}^k converge to t_m and t_{m+1}, there is a K such that $\forall k > K$, $t_m^k < s_i < t_{m+1}^k$, and thus $a_m \in b_i^*\left(s_i, x_i^k, T_{-i}^k, \mathcal{A}\right)$. Because b_i^* has a closed graph, $a_m \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$. This argument is very similar to that in the proof of Lemma 3 in Athey (2001). $\square$

Lemma 8. *If signals are affiliated; $E(C_i|s, x_i)$ is bounded, nondecreasing in s_{-i} and strictly increasing in s_i; and ties are precluded: $U_i(b', s_i, x_i|T_{-i}) \geq 0$, $U_i(b', s_i, x_i|T_{-i}) \geq U_i(b, s_i, x_i, T_{-i})$, $(s_i' - s_i)(b' - b) > 0$ imply $U_i(b', s_i', x_i|T_{-i}) \geq U_i(b, s_i', x_i|T_{-i})$.*

Proof: Consider Assumptions A.1 in Reny and Zamir (2004). A.1.i is satisfied by boundedness and continuity conditions on $E(C_i|s, x)$, A.1.ii by boundedness and risk neutrality, A.1.iii by monotonicity assumptions on the effect of s on $E(C_i|s, x_i)$ and A.1.iv by risk neutrality. Affiliation and assumptions on the joint density functions ensure that Assumption A.2 also holds. The result holds by Proposition 2.3 in that paper. $\square$

[16] For economy of notation, I will refer to the graph $\{s_i, x_i, T_{-i}, b \in [0,1] \times X_i \times \Sigma_{-i} \times \mathbb{R} : b \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})\}$ as the graph of $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ as a function of (s_i, x_i, T_{-i}).

Lemma 9. *Suppose that signals are affiliated and that $E(C_i|s,x_i)$ is bounded, nondecreasing in s_{-i} and strictly increasing in s_i. If $\forall s_i \in [\underline{s}_i, \overline{s}_i]$, $\exists a \in \mathcal{A}$ such that $U_i(a,s_i,x_i|T_{-i}) \geq 0$ then $b_i^*(s_i,x_i,T_{-i},\mathcal{A})$ is nondecreasing in the strong set order with respect to $s_i \in [\underline{s}_i, \overline{s}_i]$.*

Proof: Let $\underline{s}_i \leq s_i < s_i' \leq \overline{s}_i$, $b \in b_i^*(s_i,x_i,T_{-i},\mathcal{A})$ and $b' \in b_i^*(s_i',x_i,T_{-i},\mathcal{A})$. Let $a,a' \in \mathcal{A}$ such that $U_i(a',s_i,x_i|T_{-i}) \geq 0$ and $U_i(a',s_i',x_i|T_{-i}) \geq 0$. Suppose that $b > b'$. Notice that $(s_i'-s_i)(b'-b) < 0$. Because $b \in b_i^*(s_i,x_i,T_{-i},\mathcal{A})$

$$U_i(b,s_i,x_i|T_{-i}) \geq U_i(b',s_i,x_i|T_{-i}), \text{ and } U_i(b,s_i,x_i|T_{-i}) \geq U_i(a,s_i,x_i|T_{-i}) \geq 0.$$

Lemma 8 implies that $U_i(b,s_i',x_i|T_{-i}) \geq U_i(b',s_i',x_i|T_{-i})$ (b in this proof takes the place of b' in the lemma statement and vice versa). Thus $b \in b_i^*(s_i',x_i,T_{-i},\mathcal{A})$. Similarly, because $b' \in b_i^*(s_i',x_i,T_{-i},\mathcal{A})$

$$U_i(b',s_i',x_i,T_{-i}) \geq U_i(b,s_i',x_i,T_{-i}) \text{ and } U_i(b',s_i',x_i,T_{-i}) \geq U_i(a',s_i',x_i,T_{-i}) \geq 0$$

Lemma 8 implies that $U_i(b',s_i,x_i|T_{-i}) \geq U_i(b,s_i,x_i|T_{-i})$ (s_i in this proof takes the place of s_i' in the lemma statement and vice versa). Thus $b' \in b_i^*(s_i,x_i,T_{-i},\mathcal{A})$. It has been shown that for any $s_i < s_i'$ in $[\underline{s}_i, \overline{s}_i]$, $b \in b_i^*(s_i,x_i,T_{-i},\mathcal{A})$ and $b' \in b_i^*(s_i',x_i,T_{-i},\mathcal{A})$ implies that $\max\{b,b'\} \in b_i^*(s_i',x_i,T_{-i},\mathcal{A})$, $\min\{b,b'\} \in b_i^*(s_i,x_i,T_{-i},\mathcal{A})$. $b_i^*(s_i,x_i,T_{-i},\mathcal{A})$ is nondecreasing in s_i in the strong set order for $s_i \in [\underline{s}_i, \overline{s}_i]$. □

Fixed Point Assume that $\mathcal{A} = \mathcal{A}_i = \{a_0 < a_1 < ... < a_M\}$ for all i and that ties are broken using a priority rule. $a_M = \infty$ is equivalent to nonparticipation and $U_i(a_M,s_i,x_i,T_{-i}) = 0$ for all (s_i,x_i,T_{-i}). Fix $s \in [0,1]^n$ and $\varepsilon \in [0, 1-\max s]$. Bidders must bid a_M when they receive a signal above $1-\varepsilon$. (Athey (2001) and Reny and Zamir (2004) use the same device). $\mathcal{A}$ will be omitted from the notation in b_i^* and T_i^{BR}. Similarly, when the set of signals $[\underline{s}_i, \overline{s}_i]$ is $[0, 1-\varepsilon]$ it will be omitted from the notation in T_i^{BR}. Subsets $\mathcal{B} \subset \mathcal{A}$ and $[\underline{s}_i, \overline{s}_i] \subset [0,1]$ will not be omitted.

Define the following correspondence:

$$\begin{aligned} b_i^+(s_i,x_i,T_{-i}) &= b_i^*(s_i,x_i,T_{-i}) \text{ for all } x_i \in (\underline{x}_i, \overline{x}_i) \\ &= b_i^*(s_i,\underline{x}_i,T_{-i}) \cup \{b \in \mathcal{A} : b \leq \min b_i^*(s_i,\underline{x}_i,T_{-i})\} \text{ for } x_i = \underline{x}_i \\ &= b_i^*(s_i,\overline{x}_i,T_{-i}) \cup \{b \in \mathcal{A} : b \geq \max b_i^*(s_i,\overline{x}_i,T_{-i})\} \text{ for } x_i = \overline{x}_i \end{aligned}$$

b_i^+ is an extension of the best response correspondence that includes all high bids when $x_i = \overline{x}_i$ and all low bids when $x_i = \underline{x}_i$. b_i^+ inherits the properties of b_i^*. If the graph of $b_i^*(s_i,x_i,T_{-i})$ as a function of (x_i,T_i) is closed, the graph of $b_i^+(s_i,x_i,T_{-i})$ is also closed. If $b_i^*(s_i,x_i,T_{-i})$ is non decreasing in the strong set order in s_i, $b_i^+(s_i,x_i,T_{-i})$ is also nondecreasing.

The goal is to find a set of monotone strategies $T \in \Sigma$ and vector $x \in X$, such that T represents a set of strategies that constitute an equilibrium of the game and $t_{i,\tilde{m}} \leq s_i \leq t_{i,\tilde{m}+1}$ for all i. If $S = s$ is realized under conditions x, all bidders bid $a_{\tilde{m}}$.[17] Let $\mathcal{B}_{\tilde{m}}^- = \{a_0, a_1, ..., a_{\tilde{m}}\} \cup \{a_M\}$, and

[17]The only exception is when $t_{i,\tilde{m}} = t_{i,\tilde{m}+1}$. In this case, bidder i bids strictly below (above) $a_{\tilde{m}}$ for all signals below (above) s_i.

$\mathcal{B}_{\tilde{m}}^{+}=\{a_{\tilde{m}}, a_{\tilde{m}+1}, ..., a_M\}$.

$$\Gamma_i\left(x_i, T_{-i}\right)=\left\{\begin{array}{c}\left(w_i, y_i\right) \in\left[\underline{x}_i, \overline{x}_i\right] \times \Sigma_i: \\ \exists q:\left\{y_{i,1}, .., y_{i, \tilde{m}}, q\right\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^{-},\left[0, s_i\right]\right), \\ \left\{y_{i, \tilde{m}+1}, .., y_{i, M}\right\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^{+},\left[s_i, 1-\varepsilon\right]\right), \text{ and} \\ \min b_i^{+}\left(s_i, w_i, T_{-i}\right) \leq a_{\tilde{m}} \leq \max b_i^{+}\left(s_i, w_i, T_{-i}\right)\end{array}\right\}. \tag{57}$$

Γ_i is a correspondence that maps elements of $[\underline{x}_i, \overline{x}_i] \times \Sigma_{-i}$ to subsets of $[\underline{x}_i, \overline{x}_i] \times \Sigma_i$. Let

$$\Gamma=\left\{\Gamma_1, ..., \Gamma_n\right\}. \tag{58}$$

Γ is a correspondence that maps elements of $X \times \Sigma$ onto subsets of the same set. The following set of Lemmas shows that the conditions to apply the Kakutani Fixed point theorem hold. Lemma 13 states the properties of a fixed point of Γ.

Lemma 10. Γ *is not empty.*

Proof: By Assumption FPA.1 and Lemma 9, $b_i^*\left(\sigma_i, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^{-}\right)$ is nondecreasing in the strong set order with respect to $\sigma_i \in [0, s_i]$. It follows that $T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^{-}, [0, s_i]\right)$ is not empty. By the same argument $T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^{+}, [s_i, 1-\varepsilon]\right)$ is not empty either. Let $y_i=\{y_1, .., y_M\} \in \Sigma_i$, where $\{y_1, .., y_{\tilde{m}}, q\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^{-}, [0, s_i]\right)$ and $\{y_{\tilde{m}+1}, .., y_M\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^{+}, [s_i, 1]\right)$. Now the focus is on finding an appropriate w_i. $b_i^{+}\left(s_i, x_i, T_{-i}\right)$ is nondecreasing in the strong set order with respect to x_i; moreover, it is not empty and has a closed graph. If $a_{\tilde{m}} \leq \max b_i^*\left(s_i, \underline{x}_i, T_{-i}\right)$, then $\left(\underline{x}_i, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$. If $a_{\tilde{m}} \geq \min b_i^*\left(s_i, \overline{x}_i, T_{-i}\right)$, then $\left(\overline{x}_i, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$. If $\max b_i^*\left(s_i, \underline{x}_i, T_{-i}\right)<a_{\tilde{m}}<\min b_i^*\left(s_i, \overline{x}_i, T_{-i}\right)$, consider $w_i^1=0.5\left(\underline{x}_i+\overline{x}_i\right)$. If $\min b_i^*\left(s_i, w_i^1, T_{-i}\right) \leq a_{\tilde{m}} \leq \max b_i^*\left(s_i, w_i^1, T_{-i}\right)$, then $\left(w_i^1, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$. Instead, if $a_{\tilde{m}}<\min b_i^*\left(s_i, w_i^1, T_{-i}\right)$, let $w_i^2=0.5\left(\underline{x}_i+w_i^1\right)$ while if $\max b_i^*\left(s_i, w_i^1, T_{-i}\right)<a_{\tilde{m}}$, $w_i^2=0.5\left(w_i^1+\overline{x}_i\right)$. Repeat this procedure for w_i^2. Either this procedure eventually reaches some k such that $\min b_i^*\left(s_i, w_i^k, T_{-i}\right) \leq a_{\tilde{m}} \leq \max b_i^*\left(s_i, w_i^k, T_{-i}\right)$ and $\left(w_i^k, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$ or w_i^k converges to w_i. For all k such that $w_i<w_i^k$, $a_{\tilde{m}}<\min b_i^*\left(s_i, w_i^k, T_{-i}\right)$ whereas for all $w_i^k<w_i$, $\max b_i^*\left(s_i, w_i^k, T_{-i}\right)<a_{\tilde{m}}$. Let $\left\{w_i^{k_q}\right\}_q$ denote the subsequence such that $w_i<w_i^{k_q}$ for all q and $\left\{w_i^{k_r}\right\}_r$ denote the subsequence where $w_i^{k_r}<w_i$ for all r. By monotonicity in the strong set order $\max b_i^*\left(s_i, w_i^{k_q}, T_{-i}\right)$ converges to b^{+} and $\min b_i^*\left(s_i, w_i^{k_r}, T_{-i}\right)$ converges to b^{-}, where $b^{-}<a_{\tilde{m}}<b^{+}$. Because b_i^* has a closed graph, then $b^{+} \in b_i^*\left(s_i, w_i, T_{-i}\right)$ and $b^{-} \in b_i^*\left(s_i, w_i, T_{-i}\right)$. It follows that $\left(w_i, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$. Let $w=\{w_i\}_{i=1}^n$, and $Y=\{y_i\}_{i=1}^n$ such that $\left(w_i, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$, then $(w, Y) \in \Gamma(x, T)$. $\square$

Lemma 11. Γ *has a closed graph.*

Proof: Consider a sequence $\left(x^k, T^k, w^k, Y^k\right)$ that converges to (x, T, w, Y) such that $\left(w^k, Y^k\right) \in \Gamma\left(x^k, T^k\right)$ for all k. Consider bidder i. For all k, $\left(w_i^k, y_i^k\right) \in \Gamma_i\left(x_i^k, T_{-i}^k\right)$, and there is a q^k such that $\left\{y_{i1}^k, ..., y_{i\tilde{m}}^k, q^k\right\} \in T_i^{BR}\left(x_i^k | T_{-i}^k, \mathcal{B}_{\tilde{m}}^{-}, [0, s_i]\right)$. Take a subsequence of q^k that converges to q. $U_i\left(b, s_i, x_i | T_{-i}\right)$ is continuous in $\left(x_i, T_{-i}\right)$. By Lemmas 6 and 7, $\{y_{i1}, ..., y_{i\tilde{m}}, q\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^{-}, [0, s_i]\right)$. By the same argument, $\{y_{i\tilde{m}+1}, .., y_{iM}\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^{+}, [s_i, 1-\varepsilon]\right)$. For all k, $\min b_i^{+}\left(s_i, w_i^k, T_{-i}^k\right) \leq$

$a_{\tilde{m}} \leq \max b_i^+ \left(s_i, w_i^k, T_{-i}^k\right)$. A subsequence of $\min b_i^+ \left(s_i, w_i^k, T_{-i}^k\right)$ converges to b^- and a subsequence of $\max b_i^+ \left(s_i, w_i^k, T_{-i}^k\right)$ converges to b^+, where $b^- \leq a_{\tilde{m}} \leq b^+$. Because $b_i^+ \left(s_i, w_i, T_{-i}\right)$ has a closed graph, $b^-, b^+ \in b_i^+ \left(s_i, w_i, T_{-i}\right)$, which implies that $\min b_i^+ \left(s_i, w_i, T_{-i}\right) \leq a_m \leq \max b_i^+ \left(s_i, w_i, T_{-i}\right)$. It follows that $(w_i, y_i) \in \Gamma\left(x_i, T_{-i}\right)$ for each i, which implies that $(w, Y) \in \Gamma\left(x, T\right)$. $\square$

Lemma 12. Γ *is convex.*

Proof: Take (w_i, y_i),$(w_i', y_i') \in \Gamma_i\left(x_i, T_{-i}\right)$. Let $y_i'' = \lambda y_i + (1-\lambda)\, y_i'$ and $w_i'' = \lambda w_i + (1-\lambda)\, w_i'$ for some $\lambda \in [0,1]$.

$$
\begin{aligned}
\left\{y_{i1}, ..., y_{i\tilde{m}}, q\right\}, \left\{y_{i1}', ..., y_{i\tilde{m}}', q'\right\} &\in T_i^{BR}\left(x_i | T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right) \\
\left\{y_{i\tilde{m}}, .., y_{iM}\right\}, \left\{y_{i\tilde{m}}', .., y_{iM}'\right\} &\in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right)
\end{aligned}
$$

By Lemma 9, $b_i^*\left(\sigma_i, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-\right)$ and $b_i^*\left(\sigma_i, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+\right)$ are nondecreasing in the strong set order with respect to $\sigma_i \in [0, s_i]$ and $\sigma_i \in [s_i, 1]$, respectively. Lemma 2 in Athey (2001) ensures that both $T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right)$ and $T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right)$ are convex. Let $q'' = \lambda q + (1-\lambda)\, q'$, it follows that

$$
\begin{aligned}
\left\{y_{i1}'', ..., y_{i\tilde{m}}'', q''\right\} &\in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right) \\
\left\{y_{i\tilde{m}+1}'', .., y_{iM}''\right\} &\in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right)
\end{aligned}
$$

Without loss, assume that $w_i < w_i'' < w_i'$. It is known that

$$
\begin{aligned}
\min b_i^+\left(s_i, w_i, T_{-i}\right) &\leq a_{\tilde{m}} \leq \max b_i^+\left(s_i, w_i, T_{-i}\right) \\
\min b_i^+\left(s_i, w_i', T_{-i}\right) &\leq a_{\tilde{m}} \leq \max b_i^+\left(s_i, w_i', T_{-i}\right).
\end{aligned}
$$

By Lemma 5, $b_i^*\left(s_i, w_i, T_{-i}\right) \leq_s b_i^*\left(s_i, w_i'', T_{-i}\right) \leq_s b_i^*\left(s_i, w_i', T_{-i}\right)$. It follows that $b_i^+\left(s_i, w_i, T_{-i}\right) \leq_s b_i^+\left(s_i, w_i'', T_{-i}\right) \leq_s b_i^+\left(s_i, w_i', T_{-i}\right)$, which implies:

$$
\begin{aligned}
\min b_i^+\left(s_i, w_i'', T_{-i}\right) &\leq \min b_i^+\left(s_i, w_i', T_{-i}\right) \leq a_{\tilde{m}} \\
a_{\tilde{m}} &\leq \max b_i^+\left(s_i, w_i, T_{-i}\right) \leq \max b_i^+\left(s_i, w_i'', T_{-i}\right)
\end{aligned}
$$

Therefore, $(w_i'', y_i'') \in \Gamma_i\left(x_i, T_{-i}\right)$. Repeating the argument for all i, it follows that Γ is convex. $\square$

Lemma 13. Γ *has a fixed point. If (x, T) is a fixed point of Γ then $t_{i\tilde{m}} \leq s_i \leq t_{i\tilde{m}+1}$ for all i. Moreover, (i) if $x_i = \underline{x}_i$ then $a_{\tilde{m}} \leq \max b_i^*\left(s_i, \underline{x}_i, T_{-i}\right)$; (ii) $a_{\tilde{m}} \leq \max b_i^*\left(s_i, x_i, T_{-i}\right)$ implies that for any $\sigma_i > s_i$ such that $t_{im} < \sigma_i < t_{im+1}$, $a_m \in b_i^*\left(\sigma_i, x_i, T_{-i}\right)$; (iii) if $x_i = \overline{x}_i$ then $\min b_i^*\left(s_i, \overline{x}_i, T_{-i}\right) \leq a_{\tilde{m}}$; (iv) $\min b_i^*\left(s_i, x_i, T_{-i}\right) \leq a_{\tilde{m}}$ implies that for any $\sigma_i < s_i$ such that $t_{im} < \sigma_i < t_{im+1}$, $a_m \in b_i^*\left(\sigma_i, x_i, T_{-i}\right)$; (v) if $x_i \notin \{\underline{x}_i, \overline{x}_i\}$ then $\min b_i^*\left(s_i, x_i, T_{-i}\right) \leq a_{\tilde{m}} \leq \max b_i^*\left(s_i, x_i, T_{-i}\right)$ and $t_i \in T_i^{BR}\left(x_i | T_{-i}\right)$.*

Proof: Existence follows from Kakutani's Fixed Point Theorem. $t_{i\tilde{m}} \leq s_i \leq t_{i\tilde{m}+1}$ because $\left\{t_{i1}, .., t_{i\tilde{m}}, q\right\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right)$ and $\left\{t_{i\tilde{m}+1}, .., t_{iM}\right\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1]\right)$. (i) If $x_i = \underline{x}_i$, then $a_{\tilde{m}} \leq \max b_i^*\left(s_i, \underline{x}_i, T_{-i}\right)$. (ii) Notice the implications of $a_{\tilde{m}} \leq \max b_i^*\left(s_i, x_i, T_{-i}\right)$. Because

b_i^* is monotone in the strong order in the signal, $m > \tilde{m}$, $\sigma > s_i$ and $a_m \in b_i^*\left(\sigma, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+\right)$ imply $a_m \in b_i^*\left(\sigma, x_i, T_{-i}\right)$. Therefore, for any $\sigma > s_i$ such that $t_{im} < \sigma < t_{im+1}$, $a_m \in b_i^*\left(\sigma, x_i, T_{-i}\right)$. (iii) If $x_i = \overline{x}_i$, $\min b_i^*\left(s_i, \overline{x}_i, T_{-i}\right) \le a_{\tilde{m}}$. (iv) Notice the implications of $\min b_i^*\left(s_i, x_i, T_{-i}\right) \le a_{\tilde{m}}$. Because b_i^* is monotone in the strong order in the signal, $m < \tilde{m}$, $\sigma < s_i$ and $a_m \in b_i^*\left(\sigma, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-\right)$ imply $a_m \in b_i^*\left(\sigma, x_i, T_{-i}\right)$. Therefore, for any $\sigma < s_i$ and $m < q$ such that $t_{im} < \sigma < t_{im+1}$, $a_m \in b_i^*\left(\sigma, x_i, T_{-i}\right)$. It remains to show that if $q < \sigma < s_i$ then $a_{\tilde{m}} \in b_i^*\left(\sigma, x_i, T_{-i}\right)$. Suppose that $a_{\tilde{m}} \notin b_i^*\left(\sigma, x_i, T_{-i}\right)$. This implies that there is a $m' < \tilde{m}$, such that $a_{m'} \in b_i^*\left(\sigma, x_i, T_{-i}\right)$ and that $a_M \in b_i^*\left(\sigma, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-\right)$. It follows that $U_i\left(a_{m'}, \sigma, x_i, T_{-i}\right) = 0$. Because all bids have positive probability of winning, $a_{m'} - E\left(C_i | s_i, S_{-i} \ge S_{-i} \ge \eta_i\left(a_{m'}|T_{-i}\right), x_i\right) = 0$ which implies that $U_i\left(a_{m'}, \sigma + \varepsilon, x_i, T_{-i}\right) < 0$ and that $a_{m'} \notin b_i^*\left(\sigma + \varepsilon, x_i, T_{-i}\right)$. There is no $m' < \tilde{m}$ such that $a_{m'} \in b_i^*\left(\sigma, x_i, T_{-i}\right)$ for all σ in any subset of $[q, s_i]$. It follows that either $q = s_i$ or that $a_{\tilde{m}} \in b_i^*\left(\sigma, x_i, T_{-i}\right)$ for all $q < \sigma < s_i$. (v) If $x_i \notin \{\underline{x}_i, \overline{x}_i\}$: $\min b_i^*\left(s_i, x_i, T_{-i}\right) \le a_{\tilde{m}} \le \max b_i^*\left(s_i, x_i, T_{-i}\right)$. The results in (ii) and (iv) apply. Therefore, $t_i \in T_i^{BR}\left(x_i, T_{-i}\right)$. □

Lemma 14. *Γ has a closed graph with respect to s and ε. The fixed point also has a closed graph.*

Proof: Consider a sequence $\left(x^k, T^k, w^k, Y^k, s^k, \varepsilon^k\right) \to (x, T, w, Y, s, \varepsilon)$ where $\left(x^k, T^k\right) = \Gamma\left(w^k, Y^k | s^k, \varepsilon^k\right)$ for all k. Consider bidder i. For all k, $\left(x_i^k, t_i^k\right) \in \Gamma_i\left(w_i^k, Y_{-i}^k | s^k, \varepsilon^k\right)$ and there is a q^k such that $\left\{y_{i1}^k, ..., y_{i\tilde{m}}^k, q^k\right\} \in T_i^{BR}\left(x_i^k | T_{-i}^k, \mathcal{B}_{\tilde{m}}^-, \left[0, s_i^k\right]\right)$. Take a subsequence of q^k that converges to q. By Lemmas 6 and 7, $\{y_{i1}, ..., y_{i\tilde{m}}, q\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right)$. By the same argument, $\{y_{i\tilde{m}+1}, .., y_{iM}\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right)$. For all k, $\min b_i^+\left(s_i^k, w_i^k, T_{-i}^k\right) \le a_{\tilde{m}} \le \max b_i^+\left(s_i^k, w_i^k, T_{-i}^k\right)$. A subsequence of $\min b_i^+\left(s_i^k, w_i^k, T_{-i}^k\right)$ converges to b^- and a subsequence of $\max b_i^+\left(s_i^k, w_i^k, T_{-i}^k\right)$ converges to b^+, where $b^- \le a_{\tilde{m}} \le b^+$. Because $b_i^+\left(s_i, w_i, T_{-i}\right)$ has a closed graph, $b^-, b^+ \in b_i^+\left(s_i, w_i, T_{-i}\right)$, which implies that $\min b_i^+\left(s_i, w_i, T_{-i}\right) \le a_m \le \max b_i^+\left(s_i, w_i, T_{-i}\right)$. It follows that $(w_i, y_i) \in \Gamma\left(x_i, T_{-i} | s_i, \varepsilon\right)$ for each i, which implies that $(w, Y) \in \Gamma\left(x, T | s, \varepsilon\right)$. This proves the first part.

Consider a sequence $\left(x^n, T^n, s^n, \varepsilon^n\right) \to (x, T, s, \varepsilon)$ where $\left(x^n, T^n\right) \in \Gamma\left(x^n, T^n | s^n, \varepsilon^n\right)$ for all n. Because Γ has a closed graph in (s, ε), then $(x, T) \in \Gamma\left(x, T | s, \varepsilon\right)$. This proves the second part. □

Large Support Assumption Define $\overline{c}_i = c_i(1, ..., 1)$ and $\underline{c}_i = E\left(c_i(S) | S_i = 0, S_{-i} \ge 0\right)$. Fix some $s \in [0,1]^n$, $X = \prod_i [\underline{x}_i, \overline{x}_i]$. Let Γ^ε be the mapping defined in (57) and (58) some $\varepsilon > 0$, s^ε such that $s_i^\varepsilon = \min(s_i, 1 - 2\varepsilon)$, and $\mathcal{A} = \{a_0, a_1, ..., a_{M-1}, a_M\}$ such that $a_{\tilde{m}} \in \mathcal{A}$. Let $(x^\varepsilon, T^\varepsilon)$ be a fixed point of Γ^ε and (x, T) be the limit of some subsequence of $\left\{(x^\varepsilon, T^\varepsilon)\right\}_\varepsilon$ as $\varepsilon \to 0$. Let $\pi_i = \Pr\left(S_{-i} \ge s_{-i} | S_i = 1\right)$.

Lemma 15. *If there is a $\Delta > 0$ such that: (i) for every i, $\bar{x}_i \ge a_{\tilde{m}} + \Delta - \underline{c}_i$ and $\underline{x}_i \le a_{\tilde{m}} - \max_{j \ne i}\left(\bar{c}_j - \underline{c}_j + 4\Delta\right)\left(1 - \pi_i\right)^{-1} - \bar{c}_i$, (ii) $a_0 \le \min_i\left(\underline{c}_i + \underline{x}_i\right)$, and $\max_i\left(\overline{c}_i + \overline{x}_i\right) \le a_{M-1} < a_M = \infty$, (iii) $a_m - a_{m-1} < \Delta$ for all $m \in \{1, 2, ..., M-1\}$, and (iv) $a_{\tilde{m}} < a_{M-n}$; then $t_i \in T_i^{BR}\left(x_i | T_{-i}\right)$ and $t_{ia_M} = 1$ for all i.*

Proof: When $\varepsilon > 0$ every type of bidder makes strictly positive profits bidding $a_{M-1} \ge \max_i\left(\bar{c}_i + \overline{x}_i\right)$. Bidding infinity is suboptimal: $t_{iM}^\varepsilon = t_{iM} = 1$ for all i.

By Lemma 14, $(x,T) \in \Gamma^0(x,T)$. $x_i \notin \{\underline{x}_i, \overline{x}_i\}$ implies that $\min b_i^*(s_i, x_i, T_{-i}) \leq a_{\tilde{m}} \leq \max b_i^*(s_i, x_i, T_{-i})$, and $t_i \in T_i^{BR}(x_i|T_{-i})$ by Lemma 13. Suppose that $x_i = \overline{x}_i$. Lemma 13 ensures that $\min b_i^*(s_i, \bar{x}_i, T_{-i}) \leq a_{\tilde{m}}$. I need to show that $a_{\tilde{m}} \leq \max b_i^*(s_i, \bar{x}_i, T_{-i})$. Take any $m < \tilde{m}$.

$$U_i(a_m, s_i, \bar{x}_i|T_{-i}) = [a_m - E(c_i(S)|s_i, S_{-i} \in S_{-i} \geq \eta_i(a_m|T_{-i})) - \bar{x}_i] P(S_{-i} \geq \eta_i(a_m|T_{-i})|s_i) \leq 0$$

If $a_m \in b_i^*(s_i, \bar{x}_i, T_{-i})$ then $a_M \in b_i^*(s_i, \bar{x}_i, T_{-i})$. Therefore, $x_i = \overline{x}_i$ implies that $t_i \in T_i^{BR}(x_i|T_{-i})$. It remains to show that $t_i \in T_i^{BR}(x_i|T_{-i})$ even if $x_i = \underline{x}_i$.

Let $\phi(i)$ be such that $t^{\varepsilon}_{i\phi(i)} < 1-\varepsilon$ and $t^{\varepsilon}_{im} = 1-\varepsilon$ for all $m > \phi(i)$. Let $a^i = a_{\phi(i)}$. If $a^i > a^j$ or $a^i = a^j$ and $i > j$, then $i \succ j$. If all bidders recieve a signal arbitrarily below $1-\varepsilon$, then the winner is bidder j such that $i \succ j$ for all i. The ordering $\succ$ reflects the rank of bids: $i \succ j$ if j has a lower rank, that is, a better rank. Consider bidder j such that $j \succ i$ for all i. For all signals between $t^{\varepsilon}_{j\phi(j)}$ and $1-\varepsilon$ bidder j bids a^j and wins only when all competitors receive signals above $1-\varepsilon$. It follows that $a^j = a_{M-1}$ because otherwise bidder j could always increase the payoff when he wins without changing the set of competitors' signals where he wins. Now consider any bidder j. This bidder only wins if $S_i > 1-\varepsilon$ for all $i \prec j$ and if $S_i > t_{i,\phi(i)}$ for all $i \succ j$. Let $\tilde{a}^j$ be the maximum bid such that for all $i \succ j$, $a^i > \tilde{a}^j$ or $a^i = \tilde{a}^j$ and $i > j$. By construction $a^j \leq \tilde{a}^j$. Suppose that $a^j < \tilde{a}^j$. For all signals between $t^{\varepsilon}_{j\phi(j)}$ and $1-\varepsilon$ bidder j could increase his payoff when he wins without changing the set of competitors signals where he wins by bidding $\tilde{a}^j$. It follows that $a^j = \tilde{a}^j$. Let k be such that $i \succ k$ for all i. By backward induction, $a^k \geq a_{M-n}$. Therefore, $a^k > a_{\tilde{m}}$.

Consider the deviation by some bidder $i \neq k$ that has a signal higher than $t^{\varepsilon}_{i\phi(i)}$ to bid slightly below a^k. That bid belongs to $\mathcal{B}^+_{\tilde{m}}$ and should not be profitable if $(t_i^{\varepsilon}, x_i^{\varepsilon}) \in \Gamma_i(x_i^{\varepsilon}, T^{\varepsilon}_{-i})$. For this deviation not to be profitable to any $i \neq k$ it is necessary that $a^k - \Delta < \min_{i \neq k} \bar{c}_i + x_i^{\varepsilon}$.

Suppose that there are two bidders i and j with $\bar{c}_i + x_i^{\varepsilon}, \bar{c}_j + x_j^{\varepsilon} \leq a_{\tilde{m}} - \Delta$. Taking the minimum and using the previous result yields $a^k < a_{\tilde{m}}$. This is a contradiction. It follows that there is at most one bidder i with $\bar{c}_i + x_i^{\varepsilon} \leq a_{\tilde{m}} - \Delta$.

Suppose that $x_i^{\varepsilon} = \underline{x}_i$. By the definition of $\underline{x}_i$, $x_i^{\varepsilon} + \bar{c}_i < a_{\tilde{m}} - \Delta \leq a^k - \Delta$. Bidder i wins when all bidders receive the highest signals, i.e., $i = k$. For all $j \neq i$: $x_j^{\varepsilon} > \underline{x}_j$ and $\bar{c}_j + x_j^{\varepsilon} \geq a^k - \Delta > a_{\tilde{m}} - \Delta$. Consider two bid levels, $\underline{b}$ and $\bar{b} \in \mathcal{A}$, such that: $\min_{j \neq i}\left(\underline{c}_j + x_j^{\varepsilon}\right) - \Delta \leq \underline{b} < \min_{j \neq i}\left(\underline{c}_j + x_j^{\varepsilon}\right)$ and $a_{\tilde{m}} \leq \bar{b} \leq a^k + \Delta$. If i bids $\underline{b}$ he wins with probability one because no competitor would bid $\underline{b}$ or below. Notice that $\bar{b} \geq a_{\tilde{m}} \geq \min b_j^*\left(s_j^{\varepsilon}, x_j^{\varepsilon}, T^{\varepsilon}_{-j}\right) > \underline{b}$ for all j. The second inequality holds by Lemma 13 and $x_j > \underline{x}_j$. The third inequility follows because $\underline{b}$ is below any optimal competitors' bid. Suppose that i weakly prefers to bid $\bar{b}$ over $\underline{b}$ when he receives signal s_i.

$$\begin{aligned}
\underline{b} &\leq \bar{b}\Pr\left(S_{-i} \geq \eta^{\varepsilon}\left(\bar{b}\right)|s_i^{\varepsilon}\right) + \left(1 - \Pr\left(S_{-i} \geq \eta^{\varepsilon}\left(\bar{b}\right)|s_i^{\varepsilon}\right)\right) E\left(C_i|s_i, S_{-i} \ngeq \eta^{\varepsilon}\left(\bar{b}\right)|s_i^{\varepsilon}\right) \\
\min_{j \neq i}\left(\underline{c}_j + x_j^{\varepsilon}\right) - \Delta &< \left(\min_{j \neq i}\left(\bar{c}_j + x_j^{\varepsilon}\right) + 2\Delta\right)\Pr\left(S_{-i} \geq \eta^{\varepsilon}\left(\bar{b}\right)|s_i^{\varepsilon}\right) + \left(1 - \Pr\left(S_{-i} \geq \eta^{\varepsilon}\left(\bar{b}\right)|s_i^{\varepsilon}\right)\right)\left(\bar{c}_i + \underline{x}_i\right) \\
\min_{j \neq i}\left(\underline{c}_j + x_j^{\varepsilon}\right) - 2\Delta &< \left(\min_{j \neq i}\left(\bar{c}_j + x_j^{\varepsilon}\right) + 2\Delta\right)\pi_i + (1 - \pi_i)\left(\bar{c}_i + \underline{x}_i\right),
\end{aligned}$$

where the second line follows from the inequalities above, and the third holds for a sufficiently low

ε because $(\bar{c}_i + \underline{x}_i) < \left(\min_{j\neq i}\left(\bar{c}_j + x_j^{\varepsilon}\right) + 2\Delta\right)$ and $\pi_i \geq \Pr\left(S_{-i} \geq \eta\left(\bar{b}\right)|s_i\right)$. This implies:

$$\begin{aligned}\frac{-\left[\min_{j\neq i}\left(\bar{c}_j + x_j^{\varepsilon}\right) - \min_{j\neq i}\left(\underline{c}_j + x_j^{\varepsilon}\right) + 4\Delta\right]}{(1-\pi_i)} + \min_{j\neq i}\left(\bar{c}_j + x_j^{\varepsilon}\right) + 2\Delta &< (\bar{c}_i + \underline{x}_i) \\ \frac{-\max_{j\neq i}\left(\bar{c}_j - \underline{c}_j\right) - 4\Delta}{(1-\pi_i)} + a_{\tilde{m}} &< (\bar{c}_i + \underline{x}_i)\end{aligned}$$

This is in contradiction with the definition of $\underline{x}_i$. Therefore, the profits of an s_i^{ε}-type bidder if he bids $\underline{b}$ are positive and strictly higher than if he bids $\bar{b}$. Because $\bar{b}$ was any bid between $a_{\tilde{m}}$ and $a^i + \Delta$, monotonicity in the strong set order implies that $\max b_i^*\left(s_i^{\varepsilon}, \underline{x}_i, T_{-i}^{\varepsilon}\right) < a_{\tilde{m}}$. This contradicts Lemma 13. Therefore, $x_i^{\varepsilon} > \underline{x}_i$ and $\min b_i^*\left(s_i^{\varepsilon}, x_i^{\varepsilon}, T_{-i}^{\varepsilon}\right) \leq a_{\tilde{m}}$ for all i. Take any convergent subsequece of $\min b_i^*\left(s_i^{\varepsilon}, x_i^{\varepsilon}, T_{-i}^{\varepsilon}\right)$ and let $b^- \leq a_{\tilde{m}}$ be the limit as $\varepsilon \to 0$. Because the graph of b^* is closed (by Lemma 6), $b^- \in b_i^*(s_i, x_i, T_{-i})$. Therefore, even if $x_i = \underline{x}_i$, $\min b_i^*(s_i, x_i, T_{-i}) \leq a_{\tilde{m}} \leq \max b_i^*(s_i, x_i, T_{-i})$ and $t_i \in T_i^{BR}(x_i|T_{-i})$ (by Lemma 13). $\square$

Proof of part ii of Theorem 7 Fix x_i, $\sigma \in (0,1)^n$. Define $\bar{c}_j = c_j(1, ..., 1)$ and $\underline{c}_j = E\left(c_j(S)|S_j = 0, S_{-j} \geq 0\right)$ for every bidder j. Pick any $\Delta > 0$, and let

$$\begin{aligned}\pi_j &= \Pr\left(S_{-j} \geq \sigma_{-j}|S_j = 1\right) < 1 \\ h &= \max_j\left(\bar{c}_j - \underline{c}_j + \Delta\right)\max_j(1-\pi_j)^{-1} < \infty \\ X_j^{\sigma} &= \left[-\left(\bar{c}_j - \underline{c}_i\right) - h + x_i - 2\Delta, \left(\bar{c}_i - \underline{c}_j\right) + h + x_i + 2\Delta\right]\end{aligned}$$

Define the set X_{-i}^{σ} as the Cartesian product of X_j^{σ} for all $j \neq i$. X_{-i}^{σ} is a bounded set. Pick some $s \in [0,1]^n$ such that $s_i = \sigma_i$ and $s_{-i} \geq \sigma_{-i}$. It will be shown that there are $x_{-i} \in X_{-i}^{\sigma}$, $t \in \mathbb{R}$, and $H \in \kappa(F)$ such that if β generates H then $\left[H_{B_i|[x_i, x_{-i}]}(t)\right]_{i=1}^n = [\sigma_i, s_{-i}]$.

Let $\overline{x}_j = \bar{c}_i - \underline{c}_j + \Delta$ and $\underline{x}_j = \bar{c}_i - \max_{k\neq j}\left(\bar{c}_k - \underline{c}_k + \Delta\right)(1-\pi_j)^{-1} - \bar{c}_j$ and let $X = \prod_j\left[\underline{x}_j, \overline{x}_j\right]$. Consider an interval in the real line $\mathcal{B} = \left[\min_j\left(\underline{c}_j + \underline{x}_j\right), \max_j\left(\bar{c}_j + \overline{x}_j\right)\right]$, and a sequence of finite sets of bids $\{\mathcal{B}^q\}_q$ such that $\mathcal{B}^q \subset \mathcal{B}^{q+1} \subset \mathcal{B}$ for all q, and $\cup_{q=1}^{\infty}\mathcal{B}^q$ is dense in $\mathcal{B}$. Let

$$\mathcal{B}^1 = \left\{\min_j\left(\underline{c}_j + \underline{x}_j\right), \bar{c}_i, \bar{b}_1, ..., \bar{b}_n, \max_j\left(\bar{c}_j + \overline{x}_j\right)\right\}$$

where $\bar{c}_i < \bar{b}_1 < ... < \bar{b}_n < \max_j\left(\bar{c}_j + \overline{x}_j\right)$. $\tilde{m}^q$ is such that $a_{\tilde{m}^q} = \bar{c}_i$. Δ^q is equal to twice the maximum difference between any pair of consecutive bids in $\mathcal{B}^q$.

For each $q = 1, ..., \infty$, Let $\Gamma^{q,\varepsilon}$ be the mapping defined in (57) and (58) for s, X, set of bids $\mathcal{A}^q = \mathcal{B}^q \cup \{\infty\}$ and limit signal $\varepsilon > 0$. For all q and all consecutive $a_m, a_{m-1} \in \mathcal{B}^q$, $a_m - a_{m-1} < \Delta^q$. There exist a Q large enough so that for all $q > Q$, $\bar{x}_j \geq a_{\tilde{m}^q} + \Delta^q - \underline{c}_j$ and $\underline{x}_j \leq a_{\tilde{m}^q} - \max_{k\neq j}\left(\bar{c}_k - \underline{c}_k + 4\Delta^q\right)(1-\tilde{p}_j)^{-1} - \bar{c}_j$, where $\tilde{p}_j = \Pr\left(S_{-j} \geq s_{-j}|S_j = 1\right) \leq \pi_j$. Let $(x^{q,\varepsilon}, T^{q,\varepsilon})$ be a fixed point of the mapping $\Gamma^{q,\varepsilon}$. Let (x^q, T^q) be the limit of the sequence of fixed points as $\varepsilon \to 0$, which is also a fixed point of the mapping $\Gamma^{q,0}$ (Lemma 14). Lemma 15 shows that for all $q > Q$, T^q is an equilibrium of the discrete bid auction when the set of bids is $\mathcal{A}_i^q$ and cost shifters are x^q. Moreover, $t_{j\tilde{m}^q}^q \leq s_j \leq t_{j\tilde{m}^q+1}^q$ for all j. Let β^q denote a strategy represented by T^q. By Helley's Selection Theorem and by compactness of X, there exist a subsequence of $\{(x^q, \beta^q)\}_q$ that converges to $\left(\hat{x}, \hat{\beta}\right)$ as $q \to \infty$, where $\hat{x} \in X$ and $\hat{\beta}$ is a set of n monotone functions.

It remains to show that $\hat{\beta}$ is an equilibrium of the unrestricted game when cost shifters are $\hat{x}$. The proof of the second part of Theorem 2.1 in Reny and Zamir (2004) would apply without modifications were it not for the different treatment of the restriction that bidders with signals above $1-\varepsilon$ must bid ∞. They assume that $\varepsilon \to 0$ as $q \to \infty$, I assume that for each q, (x^q, T^q) is the limit of $(x^{q,\varepsilon}, T^{q,\varepsilon})$ as $\varepsilon \to 0$. This technical detail only changes the proof of (A.4) in that paper. I show below that the sequence (x^q, T^q) also satisfies (A.4). The rest of their proof goes through unaltered.

Take any j, pick any σ_j such that $\hat{\beta}_j(\sigma_j + v) < \infty$ for some $v > 0$. For all $q > Q$, Lemma 15 shows that $\beta_j^{q,\varepsilon}(\sigma_j) < \infty$. Because $\beta_j^{q,\varepsilon}(\sigma_j)$ wins with positive probability and $\infty \in \mathcal{A}^n$:

$$\begin{aligned} 0 &\leq \beta_j^{q,\varepsilon}(\sigma_j) - E\left(C_j|\sigma_j, S_{-j} \geq \eta_j\left(\beta_j^{q,\varepsilon}(\sigma_j)|T_{-j}^{q,\varepsilon}\right)\right) \\ &\to \beta_j^{q}(\sigma_j) - E\left(C_j|\sigma_j, S_{-j} \geq \eta_j\left(\beta_j^{q}(\sigma_j)|T_{-j}^{q}\right)\right) \\ &\leq \beta_j^{q}(\sigma_j) - E\left(C_j|\sigma_j, S_{-j} \geq \eta_j\left(\hat{\beta}_j(\sigma_j) - \delta|T_{-j}^{q}\right)\right) \\ &\to \hat{\beta}(\sigma_j) - E\left(C_j|s_j, S_{-j} \geq \eta_j\left(\hat{\beta}_j(\sigma_j)\right)|\hat{\beta}_{-j}\right). \end{aligned}$$

The second line follows after letting $\varepsilon \to 0$. There exist an ε' small enough so that $\beta_j^{q,\varepsilon}(\sigma_j) = \beta_j^q(\sigma_j)$ for all $\varepsilon < \varepsilon'$. Moreover, $T_{-j}^{q,\varepsilon} \to T_{-j}^q$ and the set $S_{-i} \geq \eta_j\left(\beta_j^{q\varepsilon}(\sigma_j)|T_{-j}^{q,\varepsilon}\right)$ converges to $S_{-i} \geq \eta_j\left(\beta_j^q(\sigma_j)|T_{-j}^q\right)$. The third line follows for any $\delta > 0$ for a sufficiently high q because the marginal cost is lower than the inframarginal. Take the limit as $q \to \infty$ for each δ such that $\hat{\beta}_j(\sigma_j) - \delta$ is not a mass point of $\min_{k \neq j} \hat{\beta}_k(S_j)$. Then take the limit as δ goes to zero to obtain the fourth line. This is (A.4) in Reny and Zamir (2004). The rest of the proof there shows that $\hat{\beta}$ is an equilibrium of the first-price auction game when bidders are allowed to bid any value in $\mathcal{A}$ and when cost shifters are $\hat{x}$.

It remains to show that in equilibrium the tie-curve passes through the vector of signals s. For each q, $t_{j\tilde{m}^q} \leq s_j \leq t_{j\tilde{m}^q+1}$. Take any $\sigma < s_j$, $\beta_j^q(\sigma) \leq a_{\tilde{m}^q}$; take any $\sigma' > s_j$, $\beta_j^q(\sigma') \geq a_{\tilde{m}^q}$. It follows that $\hat{\beta}_j(\sigma) \leq \bar{c}_i$ and $\hat{\beta}_j(\sigma') \geq \bar{c}_i$. If $\hat{\beta}$ generates H: $\left[H_{B_i|[\hat{x}_i, \hat{x}_{-i}]}(\bar{c}_i)\right]_{i=1}^n = [\sigma_i, s_{-i}]$. By Assumption FPA.3, one can add a constant to all full information costs and equilibrium bid functions to obtain a new equilibrium. Let $x_j = \hat{x}_j - \hat{x}_i + x_i$ for all j and $t = \bar{c}_i - \hat{x}_i + x_i$. Then $\beta = \hat{\beta} - \hat{x}_i + x_i$ is a Bayes Nash Equilibrium when cost shifters are $[x_{-i}, x_i]$. If β generates H: $\left[H_{B|[x_{-i}, x_i]}(t)\right]_{i=1}^n = [\sigma_i, s_{-i}]$. To complete the proof I need to show that $x_{-i} \in X_{-i}^\sigma$. By construction: $-(\bar{c}_j - \underline{c}_i) - h + x_i - 2\Delta < \underline{x}_j - \hat{x}_i + x_i$ and $\left(\bar{c}_i - \underline{c}_j\right) + h + x_i + 2\Delta > \overline{x}_j - \hat{x}_i + x_i$. Thus $x_{-i} \in X_{-i}^\sigma$. □

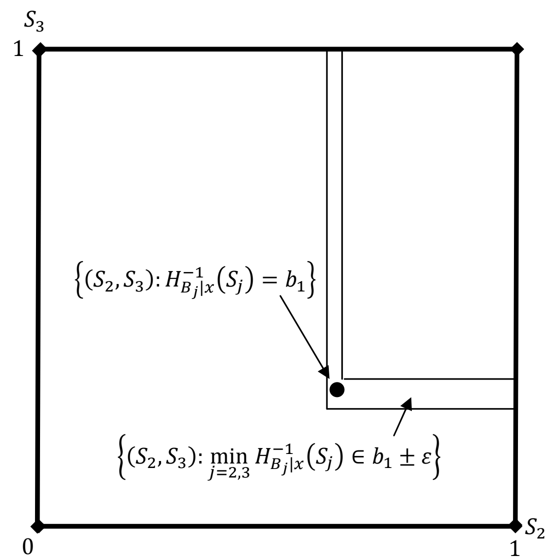

Figure 1: This figure shows the pair of competitors' signals that make them both bid exactly b_1 along with an L-shaped set containing all competitors' signals such that their minimum bid is within $\varepsilon > 0$ from b_1. The bidder's first-order optimality condition identifies the expected cost conditional on this set.

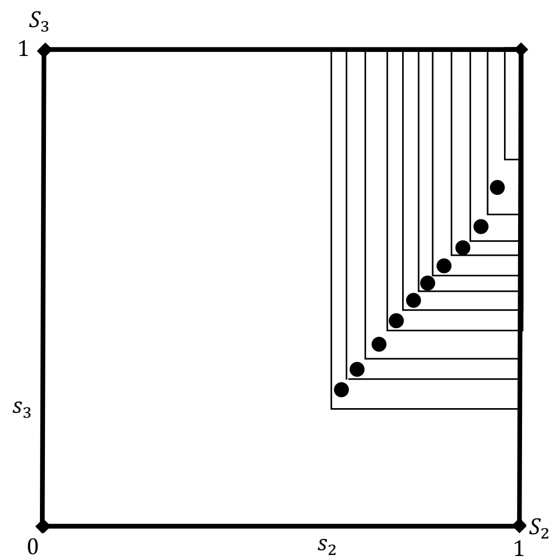

Figure 2: Variation in $[x_2, x_3, b_1]$ is used to find a different L-shaped set that stacks on top of the previous one holding s_1 and x_1 constant. The expected cost over the union of this two sets is equal to a weighted average of the expected cost over each L-shaped set. The weights are given by the probability of each set, which are identified from the joint distribution of signals. This process can be repeated to obtain a weighted average over the whole rectangle $\{S_j \geq s_j\}_{j=2,3}$.

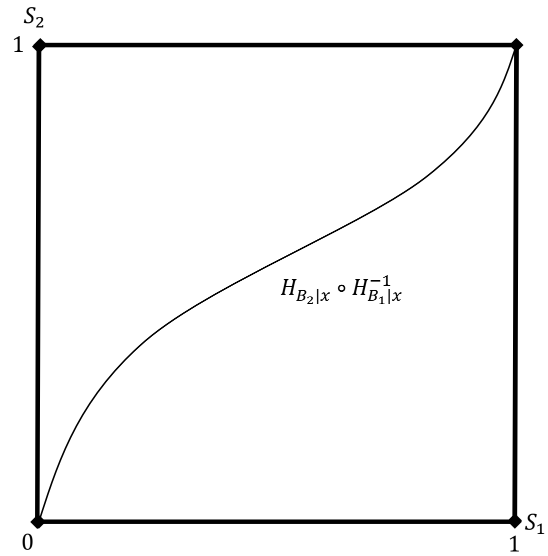

Figure 3: The tie-curve is the set of signals such that, if realized, bidders 1 and 2 tie. Formally, (s_1, s_2) in $[0,1]^2$ such that $s_1 = H_{B_1|x}(b)$ and $s_2 = H_{B_2|x}(b)$ for some $b \in \mathbb{R}$.

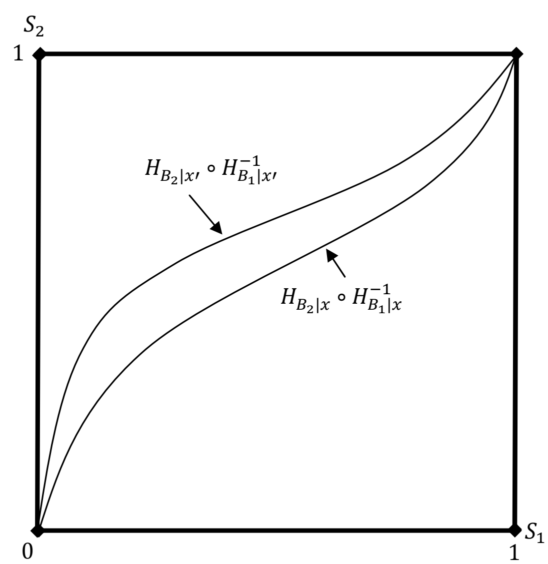

Figure 4: This figure shows the tie-curve for $x = (x_1, x_2)$, and the tie-curve for $x' = (x_1, x_2')$. Varying x_2 while x_1 is held constant shifts the tie-curve.

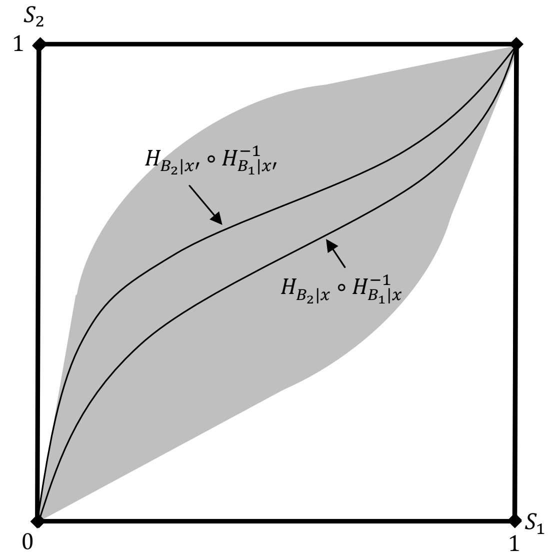

Figure 5: This figure shows the tie-curve for $x = (x_1, x_2)$, the tie-curve for $x' = (x_1, x_2')$, and the region in $[0,1]^2$ that results from drawing the tie-curves implied by other values of competitors' cost shifters. The full information cost $E(C_1|s_1, s_2, x_1)$ is identified for all (s_1, s_2) in the shaded region.

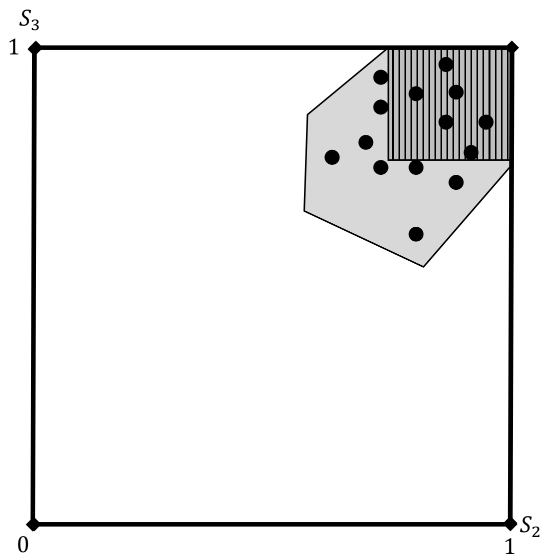

Figure 6: Consider the case with $n = 3$. Each tie-curve is a curve in the cube $[0,1]^3$. This figure shows the slice of the cube for which $S_1 = s_1$. Each tie-curve intersects this slice in one point. The figure shows several points corresponding to different tie-curves, and the region of points reachable using variation in x_2 and x_3 (shaded). The full information cost is identified in (s_1, s_{-1}, x_1) if s_{-1} is in the shaded area and if there is an integration curve originating in that point that lies within the shaded region until it reaches a point in the boundary of the square. Every curve originating in the region with vertical stripes satisfies these conditions.

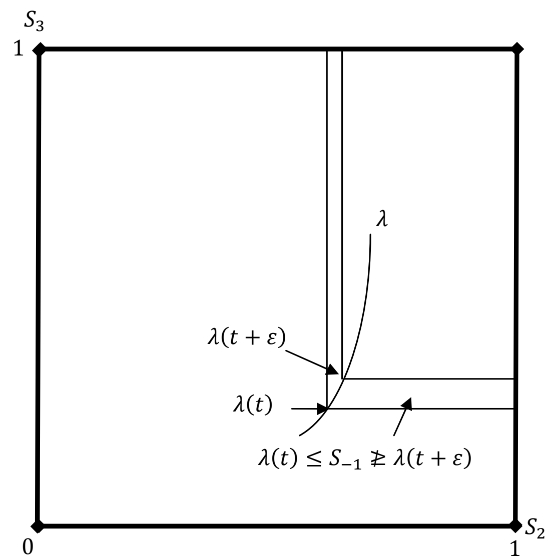

Figure 7: The pivotal set $\lambda(t) \leq S_{-i} \ngeq \lambda(t+\varepsilon)$ in the space of competitor signals.