Contemporaneous-Threshold Smooth Transition GARCH Models

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June 2009

Abstract

This paper proposes a contemporaneous-threshold smooth transition GARCH (or C-STGARCH) model for dynamic conditional heteroskedasticity. The C-STGARCH model is a generalization to second conditional moments of the contemporaneous smooth transition threshold autoregressive model of Dueker et al. (2007), in which the regime weights depend on the ex ante probability that a contemporaneous latent regime-specific variable exceeds a threshold value. A key feature of the C-STGARCH model is that its transition function depends on all the parameters of the model as well as on the data. These characteristics allow the model to account for the large persistence and regime shifts that are often observed in the conditional second moments of economic and financial time series.

Keywords: Conditional heteroskedasticity; Smooth transition GARCH; Threshold; Stock returns.

JEL Classification: C22; E31; G12.

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1 Introduction

A general class of nonlinear models of dynamic heteroskedasticity is the class of generalized autoregressive conditionally heteroskedastic (GARCH) models with regime-dependent parameters subject to smooth changes. The defining characteristic of such so-called smooth transition GARCH (STGARCH) models is that transitions between regimes are modelled by using a continuous function (usually logistic or exponential) of some observable transition variable.

Recently, Dueker et al. (2007) introduced a new class of contemporaneous-threshold smooth transition autoregressive (C-STAR) models in which the mixing (or regime) weights depend on the ex ante probabilities that regime-specific latent variables exceed certain threshold values. A key feature of the C-STAR model is that its mixing (or transition) function depends on all the parameters of the model as well as on the data, a feature which allows the model to describe time series with a wide variety of conditional distributions.

This paper contributes to the literature on nonlinear GARCH models by proposing a contemporaneous-threshold smooth transition GARCH (C-STGARCH) model motivated by the approach of Dueker et al. (2007). Our model differs from other members of the STGARCH family in two important respects. Firstly, unlike models where the argument of the mixing function used in the definition of the conditional variance is either the lagged (squared) innovation (e.g., Hagerud, 1996; González-Rivera, 1998; Lundbergh and Teräsvirta, 1998; Anderson et al., 1999; Lubrano, 2001; Madeiros and Veiga, 2009) or the lagged conditional variance (e.g., Lanne and Saikkonen, 2005), the mixing weights for C-STGARCH models are a function of both. Secondly, the mixing weights depend on all the parameters of the model. This implies that, unlike other STGARCH models, there is no need to choose an appropriate transition variable using a selection criterion since, by construction, all the variables that enter the model’s information set also enter the transition function.

The paper is organized as follows. Section 2 introduces the C-STGARCH model and discusses some of its properties. Section 3 considers maximum likelihood (ML) estimation of the parameters of the model. Section 4 presents an illustrative empirical application to U.S. stock returns. Section 5 summarizes and concludes.
2 The C-STGARCH Model

The C-STGARCH model proposed in this paper is a member of the family of STGARCH models. We say that a real-valued time series \( \{y_t\} \) follows a STGARCH(1,1) model if it satisfies the following equations:

\[
y_t - \mu = \sigma_t u_t, \quad t = 1, 2, \ldots, \tag{1}
\]
\[
\sigma_t^2 = G(z_{t-1})\sigma_{0t}^2 + \{1 - G(z_{t-1})\}\sigma_{1t}^2, \tag{2}
\]
\[
\sigma_{it}^2 = \omega_i + \alpha_i \varepsilon_{t-1}^2 + \beta_i \sigma_{t-1}^2, \quad i = 0, 1. \tag{3}
\]

In (1)–(3), \( \{u_t\} \) is a sequence of independent and identically distributed (i.i.d.) random variables with zero mean and unit variance such that \( u_t \) is independent of \( y_{t-j} \) for \( j \geq 1 \), \( G(z_{t-1}) \) is a continuous function of a vector of exogenous and/or predetermined variables \( z_{t-1} \) satisfying \( 0 \leq G(z_{t-1}) \leq 1 \), \( \varepsilon_t := y_t - \mu \), and \( \omega_i > 0, \alpha_i \geq 0, \beta_i \geq 0 \) (\( i = 0, 1 \)) and \( \mu \) are constants.

A popular choice for the mixing (or transition) function \( G \) in (2) is the logistic formulation

\[
G(s_{t-1}) = [1 + \exp(-\gamma(s_{t-1} - k))]^{-1}, \quad \gamma > 0, \tag{4}
\]

where \( s_{t-1} \) is a so-called transition variable. The location parameter \( k \) in (4) may be interpreted as the threshold between the two regimes associated with the limiting values of \( G(s_{t-1}) \) (as \( s_{t-1} \) tends to \( \pm \infty \)), while the slope parameter \( \gamma \) determines the smoothness of the transitions between the two regimes. Existing STGARCH models set \( s_{t-1} \) equal to \( \varepsilon_{t-1} \), \( \varepsilon_{t-1}^2 \) or \( \sigma_{t-1}^2 \).

To define the contemporaneous-threshold STGARCH model, assume that \( u_t^2 \) has a non-degenerate cumulative distribution function \( F \). The C-STGARCH(1,1) model is formulated by specifying the mixing function \( G \) in (2) as

\[
G(z_{t-1}) = \frac{F(k/\{\omega_0 + \alpha_0 \varepsilon_{t-1}^2 + \beta_0 \sigma_{t-1}^2\})}{F(k/\{\omega_0 + \alpha_0 \varepsilon_{t-1}^2 + \beta_0 \sigma_{t-1}^2\}) + 1 - F(k/\{\omega_1 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2\})}, \tag{5}
\]

where \( z_{t-1} = (\varepsilon_{t-1}^2, \sigma_{t-1}^2) \) and \( k \) is a non-negative threshold parameter. It is easy to see that

\[
G(z_{t-1}) = \frac{P(\varepsilon_{0t}^2 < k|z_{t-1})}{P(\varepsilon_{0t}^2 < k|z_{t-1}) + P(\varepsilon_{1t}^2 \geq k|z_{t-1})}, \tag{6}
\]
where $\epsilon^2_{it} = \sigma^2_{it}u_t^2$ $(i = 0, 1)$. Hence, under the C-STGARCH specification, (1) may be re-written as

$$
\epsilon^2_{it} = \left\{ \frac{P(\epsilon^2_{0t} < k|z_{t-1})\sigma^2_{0t} + P(\epsilon^2_{1t} \geq k|z_{t-1})\sigma^2_{1t}}{P(\epsilon^2_{0t} < k|z_{t-1}) + P(\epsilon^2_{1t} \geq k|z_{t-1})} \right\} u_t^2.
$$

Since the mixing weights are determined by the probability that the contemporaneous latent variable $\epsilon^2_{0t}$ ($\epsilon^2_{1t}$) is below (above) the threshold level $k$, we call this a contemporaneous-threshold STGARCH model.

The first-order C-STGARCH model can be straightforwardly generalized to allow for higher order dynamics by replacing the specification in (3) with

$$
\sigma^2_{it} = \omega_i + \sum_{j=1}^{q} \alpha_{ij}\epsilon^2_{t-j} + \sum_{r=1}^{p} \beta_{ir}\sigma^2_{t-r}, \quad i = 0, 1,
$$

for some $p \geq 1$ and $q \geq 1$. The mixing function of the resulting C-STGARCH($p, q$) model is defined in a way analogous to (5)–(6) with $z_{t-1} = (\epsilon^2_{t-1}, ..., \epsilon^2_{t-q}, \sigma^2_{t-1}, ..., \sigma^2_{t-p})$. For the sake of simplicity and clarity of exposition, and since the GARCH(1, 1) specification is by far the most popular in applications, we shall focus hereafter on the C-STGARCH(1, 1) model.

The C-STGARCH model differs from other models that belong to the STGARCH family in two notable respects. First, unlike models where the argument of the mixing function used in the definition of the conditional variance is $\epsilon_{t-1}$, $\epsilon^2_{t-1}$ or $\sigma^2_{t-1}$, the mixing weight in (5) is a function of both $\epsilon^2_{t-1}$ and $\sigma^2_{t-1}$. Second, the mixing weights depend on all of the model parameters. This means that for a C-STGARCH there is no need to use any selection criteria to choose the appropriate threshold variables since, by construction, all the variables that enter the information set of the model are also present in the mixing function.

3 Properties of the C-STGARCH Model

In this section, we investigate some of the key characteristics of the C-STGARCH(1, 1) model. In particular, we consider: (i) the stability of the model; (ii) the response of the mixing function to changes in the parameters of the model; and (iii) the empirical distribution of data generated by the model. In the discussion that follows, it is assumed that the parameters of the model satisfy the identification restriction \( \{\omega_0/(1 -\alpha_0 - \beta_0)\} < \{\omega_1/(1 - \alpha_1 - \beta_1)\} \), which is sufficient (but not necessary) for ensuring that \( F(k/\{\omega_0\epsilon^2_{t-1} + \beta_0\sigma^2_{t-1}\}) \) and
1 − \( F(k/\{\omega_1 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2\}) \) do not both tend to zero simultaneously. It is also assumed that \( F \) is strictly increasing and differentiable.

### 3.1 Stability of the Skeleton

As discussed in Tong (1990), one may analyze the stability properties of a nonlinear model by considering the noiseless part, or skeleton, of the model. For the C-STGARCH(1,1) model, the skeleton is defined as

\[
\bar{\varepsilon}_t^2 = S(\bar{\varepsilon}_{t-1}^2, \theta),
\]

where

\[
S(\bar{\varepsilon}_{t-1}^2, \theta) = \bar{G}(\bar{\varepsilon}_{t-1}^2)\{\omega_0 + (\alpha_0 + \beta_0)\bar{\varepsilon}_{t-1}^2\} + \{1 - \bar{G}(\bar{\varepsilon}_{t-1}^2)\}\{\omega_1 + (\alpha_1 + \beta_1)\bar{\varepsilon}_{t-1}^2\},
\]

\[
\bar{G}(\bar{\varepsilon}_{t-1}^2) = \frac{F(k/\{\omega_0 + (\alpha_0 + \beta_0)\bar{\varepsilon}_{t-1}^2\})}{F(k/\{\omega_0 + (\alpha_0 + \beta_0)\bar{\varepsilon}_{t-1}^2\}) + 1 - F(k/\{\omega_1 + (\alpha_1 + \beta_1)\bar{\varepsilon}_{t-1}^2\})},
\]

and \( \theta \) denotes all the parameters of the model. A fixed point of the skeleton is any value \( \bar{\varepsilon}_e^2 \) which satisfies the equation

\[
S(\bar{\varepsilon}_e^2, \theta) = \bar{\varepsilon}_e^2, \tag{7}
\]

and \( \bar{\varepsilon}_e^2 \) is said to be an equilibrium point of the model. Since the C-STGARCH is a nonlinear model, there may be one, several or no equilibrium points satisfying (7).

An examination of the local stability of each of the equilibrium points can be carried out by considering the following first-order Taylor expansion around the fixed point:

\[
\bar{\varepsilon}_t^2 - \bar{\varepsilon}_e^2 = S(\bar{\varepsilon}_{t-1}^2, \theta) - S(\bar{\varepsilon}_e^2, \theta) \approx \left( \frac{\partial S(\bar{\varepsilon}_{t-1}^2, \theta)}{\partial \bar{\varepsilon}_{t-1}^2} \right)_{\bar{\varepsilon}_{t-1}^2 = \bar{\varepsilon}_e^2} (\bar{\varepsilon}_{t-1}^2 - \bar{\varepsilon}_e^2). \tag{8}
\]

If the absolute value of the partial derivative in (8) is strictly less than unity, then the equilibrium is locally stable and \( \bar{\varepsilon}_e^2 \) is a contraction in the neighborhood of \( \bar{\varepsilon}_e^2 \).

It is straightforward to verify that

\[
\frac{\partial S(\bar{\varepsilon}_{t-1}^2, \theta)}{\partial \bar{\varepsilon}_{t-1}^2} = (\alpha_1 + \beta_1) + \{(\alpha_0 + \beta_0) - (\alpha_1 + \beta_1)\}\bar{G}(\bar{\varepsilon}_{t-1}^2)
\]

\[+(\omega_0 - \omega_1) + [(\alpha_0 + \beta_0) - (\alpha_1 + \beta_1)]\bar{\varepsilon}_{t-1}^2\frac{\partial \bar{G}(\bar{\varepsilon}_{t-1}^2)}{\partial \bar{\varepsilon}_{t-1}^2} \] \tag{9}
and
\[
\frac{\partial G(e^2_{t-1})}{\partial e^2_{t-1}} = -\frac{1 - F(\tau_1)}{k\{F(\tau_0) - F(\tau_1) + 1\}^2} \left\{ \frac{1}{k}ight. \frac{\partial F(\tau_0)}{\partial \tau_0} (\alpha_0 + \beta_0)^2 \tau_0^2 + \frac{1}{k} F(\tau_0) \frac{\partial F(\tau_1)}{\partial \tau_1} (\alpha_1 + \beta_1)^2 \tau_1^2, \right.
\]
where \(\tau_0 = k/\{\omega_0 + (\alpha_0 + \beta_0)e^2_{t-1}\}\) and \(\tau_1 = k/\{\omega_1 + (\alpha_1 + \beta_1)e^2_{t-1}\}\).

As a numerical illustration, consider an C-STGARCH(1,1) model with \(u_t \sim N(0, 1)\) and the following parameter configuration:
\[
\mu = 0.3, \quad (\omega_0, \omega_1) = (0.01, 0.02), \quad (\alpha_0, \alpha_1) = (0.51, 0.1), \\
(\beta_0, \beta_1) = (0.40, 0.75), \quad k = 1.
\]
(11)
We use a grid of starting values to solve equation (7) numerically and find the number of equilibrium points; the local stability of each equilibrium point is then assessed by considering the expansion in (8)–(10). A single equilibrium point \(e^2_t = 0.112\) is found, with the associated partial derivative in (8) being equal to 0.942, suggesting that the model is locally stable.

3.2 Properties of the Mixing Function

As mentioned before, a key feature of the C-STGARCH model is that its mixing function \(G\) depends on all the parameters of the model as well as on both \(e^2_{t-1}\) and \(\sigma^2_{t-1}\). The signs of \(\partial G/\partial \alpha_0\) and \(\partial G/\partial \beta_0\) are both negative. An increase in \(\alpha_0\) and/or \(\beta_0\) raises \(\omega_0 + \alpha_0 e^2_{t-1} + \beta_0 \sigma^2_{t-1}\) and reduces the probability \(P(e^2_{t|t} < k|z_{t-1})\) and thus \(G(z_{t-1})\). A similar argument applies for a change in \(\alpha_1\) and/or \(\beta_1\), with the signs of both \(\partial G/\partial \alpha_1\) and \(\partial G/\partial \beta_1\) being negative. A change in \(\alpha_1\) and/or \(\beta_1\) raises \(\omega_1 + \alpha_1 e^2_{t-1} + \beta_1 \sigma^2_{t-1}\), increases the probability \(P(e^2_{t|t} > k|z_{t-1})\), thus reducing \(G(z_{t-1})\).

The sign of \(\partial G/\partial k\) is always positive since the higher the threshold is the bigger is the area of the conditional density of \(e^2_{t|t}\) which is below the threshold and the smaller is the area of the conditional density of \(e^2_{t|t}\) which is above the threshold. In other words, an increase in \(k\) results in an increase in \(F(k/\{\omega_0 + \alpha_0 e^2_{t-1} + \beta_0 \sigma^2_{t-1}\})\) and a decrease in \(1 - F(k/\{\omega_1 + \alpha_1 e^2_{t-1} + \beta_1 \sigma^2_{t-1}\})\). The sign of \(\partial G/\partial \omega_1\) is always negative since \(1 - F(k/\{\omega_1 + \alpha_1 e^2_{t-1} + \beta_1 \sigma^2_{t-1}\})\) is higher the larger \(\omega_1\) is. Analogously, the sign of \(\partial G/\partial \omega_0\) is always negative. Note also that the signs of \(\partial G/\partial e^2_{t-1}\) and \(\partial G/\partial \sigma^2_{t-1}\) are negative.
3.3 Empirical Distribution of the Data

There is a large variety of empirical distributions and time series that can be generated by the C-STGARCH model. In Figure 1, we show the conditional state-dependent distributions (for two different conditioning values), the threshold, the histogram of $|\varepsilon_t|$, and the time series of $\varepsilon_t^2$ and $G(z_{t-1})$ generated by a C-STGARCH$(1,1)$ model. We used 400 realizations for the time-series evolution of $\varepsilon_t$ and $G(z_{t-1})$, with the parameter values given in (11) and $u_t$ having Student’s $t$-distribution with 3 degrees of freedom (rescaled to have unit variance).

The upper left panel of Figure 1 shows the state dependent on $z_{t-1} = (0.26, 0.137)$ (observation 80 of the simulated data). For this conditioning value, the value of the mixing function is 0.98. It can be seen in the lower right panel that it coincides with relatively very low values of the conditional variance and therefore it assigns a high weight to the model which is associated with values smaller than the threshold.

The upper right panel of Figure 1 shows the state dependent on $z_{t-1} = (3.74, 8.28)$ (observation 125 of the simulated data). For this value of $z_{t-1}$, the mixing function is equal to 0.47. It can be seen in the lower right panel that it coincides with relatively very high values of the conditional variance and therefore it assigns a high weight to the model which is associated with values smaller than the threshold.

The lower left panel shows the histogram of the generated data, which reveals that most of the observations with high probability of regime 0 with values between 0 and 1 while those with high probability of regime 1 with values greater than 1. We can see from the histogram that most of the observations in the sample are associated with high probability of coming from regime 1.

Finally in the lower right panel we can see that $G(z_{t-1})$ takes very low values for periods of very high volatility and that the skeleton of the model converges very rapidly to the equilibrium point $\tilde{\varepsilon}_t^2 = 0.111$. We found that the value of the partial derivative in (8) is 0.9524, suggesting that this equilibrium is locally stable.

4 Parameter Estimation

Once the probability distribution of $u_t$ in (1) is specified, the parameters of a C-STGARCH model can be estimated straightforwardly by the ML method. Letting $f_u$ denote the prob-
ability density function of \( u_t \), the log-likelihood function of a sample \( \{y_1, \ldots, y_T\} \) from the C-STGARCH\((1, 1)\) model (ignoring initial conditions) is

\[
L(\theta) = \sum_{t=1}^{T} \{- \ln \sigma_t + \ln f_u(\varepsilon_t / \sigma_t)\},
\]

where

\[
\sigma_t^2 = G(z_{t-1})(\omega_0 + \alpha_0 \varepsilon_{t-1}^2 + \beta_0 \sigma_{t-1}^2) + \{1 - G(z_{t-1})\}(\omega_1 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2),
\]

\( G(z_{t-1}) \) is given by (5), \( \theta = (\mu, \omega_0, \alpha_0, \beta_0, \omega_1, \alpha_1, \beta_1, k, \lambda) \), and \( \lambda \) is a vector of (unknown) shape parameters that specify \( f_u \).

In the simulations and empirical application that follow, \( f_u \) is specified to be the probability density function of Student’s \( t \)-distribution with \( m \) degrees of freedom (rescaled to have unit variance), so that

\[
f_u(x) = \frac{\Gamma\left(\frac{\{m + 1\}/2}{m/2}\right)}{\sqrt{\pi(m-2)}} \left(1 + \frac{x^2}{m} \right)^{-\frac{1}{2}(1+m)/2}, \quad -\infty < x < \infty, \quad m > 2, \tag{12}
\]

where \( \Gamma(\cdot) \) is Euler’s gamma function. ML estimation based on the \( t \)-distribution is arguably more appropriate than estimation based on a Gaussian likelihood since many financial time series exhibit substantial leptokurtosis which may not be adequately accounted for by conditional heteroskedasticity alone (see, e.g., Bollerslev, 1987). Note that, under the maintained assumption in (12), the values of \( F \) in (5) can be computed as

\[
F(x) = H\left(\frac{mx}{m-2}\right), \quad x > 0,
\]

where \( H \) is the cumulative distribution function of the central \( F \)-distribution with 1 and \( m \) degrees of freedom, i.e.,

\[
H(x) = \frac{\Gamma\left(\frac{\{m + 1\}/2}{m/2}\right)}{\sqrt{\pi m}} \int_{0}^{x} \frac{1}{\sqrt{z}} \left(1 + \frac{z}{m} \right)^{-\frac{1}{2}(1+m)/2} \, dz, \quad x > 0.
\]

The asymptotic properties of the ML estimator are currently unknown for our model. However, if a C-STGARCH model satisfies suitable stationarity, ergodicity and identifiability conditions, it is reasonable to expect that standard asymptotic results for statistical inference (e.g., Crowder, 1976) apply.

To throw some light on the sampling properties of the maximum likelihood estimator of the parameters of a C-STGARCH\((1, 1)\), we now discuss the results of a simulation study. The
data-generating process in the sampling experiments is the model defined by (1)–(5), with \{u_t\} being i.i.d. random variables having the probability density function (12) with \(m = 3\). The experiments are a full factorial design of:

\[
\begin{align*}
\mu &= 0.3, \\
(\omega_0, \omega_1) &\in \{(0.005, 0.01), (0.01, 0.02), (0.02, 0.04), (0.05, 0.12)\}, \\
(\alpha_0, \alpha_1) &\in \{(0.51, 0.1), (0.25, 0.10), (0.25, 0.05)\}, \\
(\beta_0, \beta_1) &\in \{(0.40, 0.75), (0.74, 0.60), (0.60, 0.94)\}, \\
k &\in \{0.2, 0.3, 0.4, 0.5, 0.6, 1\}, \\
T &\in \{100, 200, 400, 800, 1600\}.
\end{align*}
\]

In each Monte Carlo replication, \(50 + T\) data points for \(y_t\) are generated with \(y_0^2 = \sigma_0^2 = k\), but only the last \(T\) of these are used in order to attenuate the effect of the starting values. The ML estimate \(\hat{\theta} = (\hat{\mu}, \hat{\omega}_0, \hat{\omega}_1, \hat{\alpha}_1, \hat{\beta}_1, \hat{k}, \hat{m})\) is obtained by means of a quasi-Newton algorithm that approximates the Hessian according to the Broyden–Fletcher–Goldfarb–Shanno (BFGS) update computed from numerical derivatives. For each design point, a grid of 7 values for each parameter (including the true value) are used as starting values for the BFGS iterations; the starting values that result in the largest likelihood are then selected.\(^1\) Finally, since the computation of ML estimates is particularly time consuming (given the large number of design points and the grid for initial values), the number of Monte Carlo replications per experiment is 2,000.

In order to save space, only a selection of simulation results are reported here.\(^2\) More specifically, we focus on the following parameter configurations, in which the threshold \(k\) is allowed to vary:

\[
\begin{align*}
\mu &= 0.3, \quad (\omega_0, \omega_1) = (0.01, 0.02), \quad (\alpha_0, \alpha_1) = (0.51, 0.1), \quad (\beta_0, \beta_1) = (0.40, 0.75), \\
k &\in \{0.3, 0.4, 0.6, 1\}.
\end{align*}
\]

Table 1 records the finite-sample bias of the ML estimator of \(\theta\). The results show that the ML estimator tends to be somewhat biased for the smaller sample sizes under consideration.

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\(^1\)It is worth noting that estimation results were found to be robust with respect to the choice of initial values.

\(^2\)The full set of results is available on request.
but the bias clearly decreases as the sample size increases. As a measure of the accuracy of estimated asymptotic standard errors as approximations to the sampling standard deviation of the ML estimator, Table 2 shows the ratio of the exact standard deviation of the ML estimates to the estimated standard errors averaged across replications for each design point. The standard errors are calculated in the familiar manner from the inverse of the Hessian of the log-likelihood function evaluated at the ML estimates. For the vast majority of cases, the estimated asymptotic standard errors are downward biased. These biases are not, however, substantial (at least for samples sizes exceeding 400) and should not have significant adverse effects on inference.

5 Empirical Application

In this section, we illustrate the practical use of the proposed C-STGARCH model using a time series of U.S. daily stock returns. Our data set consists of continuously compounded daily returns of the S&P 500 index over the period January 1, 1964 to March 12, 2007. The returns are pre-filtered by means of a first-order autoregressive model to remove serial correlation.

The C-STGARCH$^{(1,1)}$ model defined in (1)–(3) is compared to the following models:

(i) GARCH:

$$\sigma_t^2 = \omega + \alpha_0 \varepsilon_{t-1}^2 + \beta_0 \sigma_{t-1}^2,$$

(ii) STGARCH(a):

$$\sigma_t^2 = \omega + [\alpha_0 G(\sigma_{t-1}^2) + \alpha_1 (1 - G(\sigma_{t-1}^2))]\varepsilon_{t-1}^2 + \beta_0 \sigma_{t-1}^2,$$

(iii) STGARCH(b):

$$\sigma_t^2 = \omega + \alpha_0 \varepsilon_{t-1}^2 + [\beta_0 G(\sigma_{t-1}^2) + \beta_1 (1 - G(\sigma_{t-1}^2))]\sigma_{t-1}^2,$$

(iv) STGARCH(c):

$$\sigma_t^2 = \omega + (\alpha_0 \varepsilon_{t-1}^2 + \beta_0 \sigma_{t-1}^2)G(\sigma_{t-1}^2) + (\alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2)\{1 - G(\sigma_{t-1}^2)\},$$

where

$$G(\sigma_{t-1}^2) = \left[1 + \exp(-\gamma(\sigma_{t-1}^2 - k))\right]^{-1}, \quad \gamma > 0.$$
Note that, in all nonlinear models, the constant term in the volatility equation (which determines the level of the conditional variance) is assumed to be time-invariant. This restriction is imposed in order to avoid associating one of the regimes with the outlier observations that are present in our sample.

The ML estimates of the parameters of the five volatility models (assuming \( t \)-distributed innovations) are shown in Table 3, together with corresponding asymptotic standard errors. We also report the value of the Ljung and Box (1978) portmanteau statistic (\( Q_b \)) based on the first \( b = 35 \) and \( b = 45 \) sample autocorrelations of the squared standardized residuals, the value of the maximized log-likelihood (\( \mathcal{L}_{\text{max}} \)), and the value of the Akaike information criterion (AIC).

The parameter estimates of the GARCH model are consistent with the literature in that the fitted model implies strong persistence in the variance as measured by the estimate of the sum \( \alpha_0 + \beta_0 \). For the three logistic STGARCH models, the estimated threshold parameter \( (k) \) varies between 1.46 and 6.61. The STGARCH(a) model has an estimated adjustment parameter \( (\gamma) \) with a large standard error and the estimated mixing weights plotted in Figure 2 do not show marked movement. The STGARCH(b) model exhibits signs of misspecification and has a large adjustment parameter \( (\hat{\gamma} = 6.61) \). This is probably due to the fact that the smooth-transition mechanism of the model essentially captures outliers in the squared returns, as can be seen from the estimated mixing weights shown in Figure 3. The STGARCH(c) model appears to be the most successful of the three logistic STGARCH models.

The estimated parameters of the C-STGARCH model reveal remarkably different behaviour in the two regimes. The response to the lagged squared shock is much more substantial in regime 1 (\( \hat{\alpha}_1 = 0.18 \)) than in regime 0 (\( \hat{\alpha}_0 = 0.03 \)). This in turn implies that big shocks are amplified in regime 1, which is therefore a regime associated with periods of high conditional volatility. Nevertheless, the estimated regime-specific persistence parameter is greater for regime 0 than for regime 1 (\( \hat{\beta}_0 = 0.96 \) and \( \hat{\beta}_1 = 0.81 \), respectively). The stability of the empirical C-STGARCH model is assessed by numerical simulation. The skeleton of the model is found to have a single fixed point \( \bar{\varepsilon}^2_{t-1} = 0.345 \). The derivative \( \partial S(\bar{\varepsilon}^2_{t-1}, \vartheta) / \partial \bar{\varepsilon}^2_{t-1} \) in (9) is 0.9856 when evaluated at \( \bar{\varepsilon}^2_{t-1} = \bar{\varepsilon}^2_t \), suggesting that the empirical model is locally stable. The values of the mixing function shown in Figure 5 suggest that the regimes are highly persistent.
and that the separation mostly associates periods of high conditional volatility with regime 1. The separation of regimes is quite similar to that implied by the STGARCH(c) model, as can be seen in Figure 4.3.

Finally, the models in Table 3 are compared in terms of two additional criteria, namely the mean square error loss \( \text{MSE} = \frac{1}{T} \sum_{t=1}^{T} (\hat{\epsilon}_t^2 - \hat{\sigma}_t^2)^2 \) and the coefficient of determination \( R^2 \) in the regression of \( \ln \hat{\epsilon}_t^2 \) on \( \ln \hat{\sigma}_t^2 \) and a constant (cf. Pagan and Schwert, 1990), where \( \hat{\epsilon}_t \) and \( \hat{\sigma}_t^2 \) are the residuals and the estimated conditional variance, respectively. The MSE and \( R^2 \) criteria favour the C-STGARCH model, as does the AIC.

6 Summary

In this paper, we have proposed a contemporaneous-threshold autoregressive conditionally heteroskedastic model, which may be thought of as a special case of the STGARCH model. A key feature of the C-STGARCH is that its transition function depends on all the parameters of the model as well as on the data. These characteristics allow the model to account for the large persistence and regime shifts that are often found in the conditional second moments of many economic and financial time series. We have discussed the properties of the model and evaluated the finite-sample properties of the ML estimator of its parameters. We have also presented an empirical application to the daily returns on the S&P 500 index, which has shown that the proposed C-STGARCH model is capable of outperforming some competing nonlinear GARCH models.

References


\(^3\)Note that the labeling of regimes as 0 and 1 for the C-STGARCH model is the reverse of that for the logistic STGARCH models.


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[0.9536] [0.9384] [0.0000] [0.5230] [0.8925]

$Q_{45}$ 29.2586 30.0474 127.8812 40.8075 32.8151

[0.9666] [0.9575] [0.0000] [0.6501] [0.9116]

$L_{\text{max}}$ –13057.4 –13053.3 –13054.1 –13051.7 –13043.7

AIC 26124.8 26122.2 26124.2 26121.4 26103.4

MSE 4.6082 4.7541 5.1725 4.6096 4.6045

$R^2$ 0.0786 0.0792 0.0510 0.0794 0.0800

Figures in parentheses are asymptotic standard errors. Figures in square brackets are asymptotic $p$-values.
Distributions, Generated Data and Skeleton for DGP1

Regime Specific \( F \)-Distributions Conditional on \( \xi_{t-1} = 0.25 \) and \( \sigma_{\xi} = 0.137 \), \( G(x_{t-1}) = 0.96 \), \( t = 80 \) of the generated data.

Regime Specific \( F \)-Distributions Conditional on \( \xi_{t} = 3.74 \) and \( \sigma_{\xi} = 8.28 \), \( G(x_{t-1}) = 0.47 \), \( t = 125 \) of the generated data.

Histogram of the Data Generated by the Model

Generated Data, Threshold and Mixing Function

Figure 1
CRSP Returns and Mixing Function for the STGARCH(\(a\)) Model

CRSP Absolute Value of Daily Returns

Mixing Function

Figure 2
CRSP Returns and Mixing Function for the STGARCH(b) Model

Figure 3
CRSP Returns and Mixing Function for the STGARCH(ν) Model

CRSP Absolute Value of Daily Returns

Mixing Function

Figure 4
CRSP Returns and Mixing Function for the C-STGARCH

CRSP Absolute Value of the Daily Returns

CRSP Threshold and Skeleton

Mixing Function

Figure 5