Introduction

This paper analyses the effects of relaxing one of the critical underlying assumptions of the textbook approach to investment under uncertainty for partial equilibrium models. Most textbook models assume that either the potential investor has access to a single project or she can consider competing (or complementary) projects independently (for example, Dixit and Pyndick (1994)). This paper investigates the effect of relaxing these assumptions.

For a multi-product monopolist, the decision on producing a new good (such as the introduction of new financial instruments, new software, etc.) will typically affect the profits obtained from the existing good. Then, conditional on having first invested in the production of one good, the optimal decision on producing an additional good will crucially be affected by the nature of the goods that are produced. Furthermore, once we allow the cashflows from the products to be interrelated and consider the decision problem ex-ante, we find that, in general, not only is the standard rule for optimal investment not valid, but also the sequence of investment decisions might be different from the textbook rule.

More particularly, in the case of two goods, we find that the optimal entry thresholds depend crucially on the degree of substitutability or complementarity between the goods. When the goods are substitutes, the investment strategy is usually sequential and the entry threshold for the second good cannot be derived using the textbook approach, that is, without taking into consideration the parameters linking the demand functions of the two goods. When the goods are complements, depending on the degree of complementarity, the investment strategy may be simultaneous or sequential. We also find that when goods are extreme substitutes in the sense that the demand for one completely wipes out the demand for the other, it might be optimal for the monopolist to first invest in one good, and then at some later date exit the production of this good and enter the production of the other good (even though there is no option value to exit).

In section 2 we present our model. The textbook case of isolated investment projects is illustrated in section 3. In section 4 we consider the investment strategies open to the monopolist and the timing of investment. In section 5 we analyze the problem of choosing the optimal investment strategy. We summarize our results in the final section.

The Model

Consider a monopolist who has the possibility of investing in the production of goods $A$ and $B$. The inverse demand function for good $i = A, B$ is given by

$$ P_i = YD_i + Q_i \bar{Y}_i, \quad j = A, B, j \in E_i, $$

where $D_i$ is differentiable. $Y$ is a multiplicative shock which follows a geometric Brownian motion

$$ dY = \bar{Y} dt + \bar{a} Y dw, $$

where $dw$ is the increment of a Wiener process with $E dw = 0$ and $E dw^2 = dt$. Notice that for this demand function, the (instantaneous) quantity produced is deterministic while the price is stochastic. Recall that in the inverse demand function, a positive (negative) $\frac{\partial D_i}{\partial Q_j}$, $i \in E_j$, implies that the two products are complements (substitutes), and $\frac{\partial D_i}{\partial Q_j} = 0$, $i \in E_j$, implies that the two goods are independent of each other. We also assume that the variable costs of production are zero for both goods, and investing in each good requires a fixed and fully irreversible investment cost $I_i > 0$.

Independent Products

In this section we consider the textbook case of independent products where the profit stream of each product is independent of the other product. In our framework this corresponds to the case of looking at the investment decision problem of each good by assuming that the other is fixed at zero. Thus, we examine the investment decision problem of each good $i$ by setting $Q_j = 0$, $j \in E_i$, that is, we use $P_i = YD_i + Q_i \bar{Y}_i$ as the inverse demand function of good $i$.

Conditional on the independence of the profit streams, let $F_i = \bar{Y}_i$ be the value of the option to invest in good $i$, and let $V_i = \bar{Y}_i$ be the value of the project consisting in the production of good $i$, where $\bar{Y}_i$ is the corresponding instantaneous cashflow. Using the notation $L^0 \bigcup \max D_i Q_i 0 \bar{Y}_i$, the instantaneous cashflow can be written as $\bar{Y}_i = L^0 \bar{Y}_i$. The value of the option to invest is found by solving the Bellman’s equation: $F_i = \bar{Y}_i$ where $\bar{Y}_i$ is the monopolist’s discount factor. Applying Ito’s lemma, this reduces to $\bar{Y}_i = (1/2) \bar{a}^2 \bar{Y}_i^2 + \bar{a} \bar{Y}_i$. $F_i = 0$, which has the following solution

$$ F_i = \bar{Y}_i = C_1 \bar{Y}_1 + C_2 \bar{Y}_2, $$

where $C_1$ and $C_2$ are constants to be determined using the boundary conditions, $K_1 > 1$ and $K_2 > 0$. The value of the project consisting in producing good $i$ (the discounted profits stream) is $V_i = \bar{Y}_i$.
The usual boundary conditions are $F_i > 0 \iff \gamma_0 = 0$, $F_i > \gamma_0 \iff V_i > \gamma_0$, and $F_i > 0 \iff \gamma_0$. These boundary conditions give the following entry threshold for good $i$: $\gamma_0 = \frac{K_i}{\ln(1 + K_i)}$ if $V_i > \gamma_0$. Using $\gamma_1 = L_Y V_i$, the entry threshold for good $i$ can be alternatively written as

$$y_1 = \left[\frac{K_i}{L_y K_i} \ln(1 + K_i)\right] \iff V_i > \gamma_0.$$  

From the optimal entry threshold in (3), we can show that the expected time of entry to produce good $i$ is given by:

$$E[T_i^0] = \frac{\ln(y_1^0)}{2a^2/\alpha}.$$  

The textbook investment rule as given by $\gamma_3 \iff \gamma_4 \iff \gamma_5$ has the interpretation that the investment is postponed, the larger is the variance of the stochastic process, the smaller is the drift parameter, the larger is the investment cost, the larger is the monopolist’s discount factor and the lower the demand for its product.

In the rest of the paper, we normalize investment and demand parameters so that $I_A = I_B$ because of $\gamma_4$. So, using the textbook approach, our normalization implies that, whenever $Y < Y_B$, entry in the production of good $A$ takes place first and entry in the production of good $B$ takes place later.

**Investment strategies and thresholds**

**Sequential Investment**

In this section we solve the monopolist’s entry thresholds for the case of sequential investment in two steps. In the first step we consider the entry threshold for the production of good $i = A, B$ assuming that the monopolist has already entered the production of good $i = A, B / E$ in the second step we consider the entry threshold for the production of good $i$ when there is an option to also invest in the production of good $j$.

**Step 1**

Let $F_i > \gamma_0 \iff \gamma_0$ be the value of the option to enter in the production of $j$ conditional on the monopolist having invested in the production of good $i$, and let $V_i > \gamma_0 \iff \gamma_0$ be the present discounted value of the monopolist’s future cashflow, where $\gamma_0$ is the instantaneous cashflow of the combined production of the two goods after investing in good $j$ conditional on having invested in good $i$ footnote Using the notation $\hat{R}_0 \iff \gamma_0$ can be written as $\gamma_0 = \hat{R}_0$. Footnote At the critical profit level, the monopolist gives up the option to invest in the production of $i \iff V_i \iff \gamma_0$ because of $\gamma_3$. The present discounted value of the cashflow from the production of good $i$, and incurs an additional investment cost $I_j$. In exchange the monopolist obtains the present discounted value of the cashflows from the combined production of the two goods. Thus, the corresponding value matching condition at the entry threshold is given by $F_i > \gamma_0 \iff \gamma_0$.

By using standard techniques as in section 3, and substituting for $\gamma_i$ and $\gamma_0$, the value matching condition becomes

$$C_3 \iff \gamma_0 \iff + \left(\frac{L^D}{-R} \right) \iff + I_j = \left(\frac{\hat{R}_0}{-R} \right).$$  

where $C_3$ is a constant to be determined using the boundary conditions. The smooth pasting condition at $\gamma_0$ is given by

$$K_i \iff \hat{R}_3 \iff \gamma_0 \iff + \left(\frac{L^D}{-R} \right) = \left(\frac{\hat{R}_0}{-R} \right).$$  

Then, conditional on having already invested in the production of good $i$, $\gamma_3 \iff \gamma_4 \iff \gamma_5$ imply the following entry threshold for investment in the production of good $j$,

$$\gamma_j = \left[\frac{K_j}{L_y K_j} \ln(1 + K_j)\right] \iff V_i > \gamma_0.$$  

Therefore, conditional on having already invested in good $i$, the expected entry time for investment in good $j$ is given by
Step 2

Next, we consider the problem of investing in the production of good \( i \), knowing that investment in the production of good \( j \) (\( j \neq i \)) could also be undertaken at a later time. Let \( F_i^* \) be the value of the option to invest in the production of good \( i \), conditional on the assumption that the monopolist can also invest in the production of good \( j \) in the future (i.e. produce both goods in the future). At the critical profits level, the monopolist gives up the option to enter the production of good \( i \) and incurs the investment cost \( I_i \). In exchange, the monopolist receives the present discounted value of the cashflows from the production of good \( i \), and the value of the option to invest in \( i \) in \( i \) time. Thus, the corresponding value matching condition at the entry threshold is given by \( F_i^* + I_i = V_i^* + F_{Mj^*}^* \). Then, by using standard techniques and substituting for \( F_i^* \) and \( F_{Mj^*}^* \), which are defined as before, the value matching condition becomes

\[
C_4 F_i L_i^0 C_4 L_i^0 R_i^* K_i^* + I_i = \left[ \ln \left( \frac{F_i}{L_i^0 K_i^*} \right) \right] + C_3 F_i R_i^* K_i^*.
\]

where \( C_4 \) is another constant to be determined using the boundary conditions. The smooth pasting condition is

\[
K_i L_i^0 C_4 L_i^0 R_i^* K_i^* = \left[ \ln \left( \frac{D_i}{L_i^0 K_i^*} \right) \right] + K_i R_i^* K_i^*.
\]

Solving \( 9f_i \) and \( 10f_i \) we get the following entry threshold for undertaking the production of good \( i \), conditional on the assumption that investment in the production of good \( j \) may also be undertaken at a later time

\[
\hat{Y}_i = \left[ \frac{K_i}{L_i^0 K_i^*} \right] > \frac{W_i}{I_i^*}.
\]

The corresponding expected entry time is given by

\[
E^{T_i} F_i = \left[ \ln \left( \frac{\hat{Y}_i}{W_i} \right) \right] = \left[ \ln \left( \frac{\frac{K_i}{L_i^0 K_i^*}}{W_i} \right) \right] = \frac{W_i}{I_i^*}.
\]

The analysis of this section tells us that it is feasible to invest sequentially in good \( i \) first and good \( j \) later only if the thresholds in \( 7f_i \) and \( 11f_i \) are consistent in the sense that \( \hat{Y}_i < \hat{Y}_j < K \), which holds if and only if \( \frac{K_i}{K_j} > \frac{L_i^0}{L_j^0} \). Furthermore, we can also say that, whenever it is feasible to invest sequentially in good \( i \) first and good \( j \) later, if investment is to be undertaken in this sequence, then the appropriate thresholds are those given in \( 7f_i \) and \( 11f_i \). Thus, in this section we have examined only the issue of optimal timing of entry conditional on a given sequence of investment, and not whether a given sequence of investment is optimal or not.

Note that the entry threshold in \( 7f_i \) is different from the entry threshold in the textbook case (i.e. \( Y_i = 0 \)), and the entry threshold in \( 11f_i \) is exactly the same as the entry threshold in the textbook case (i.e. \( Y_i = 0 \)). Thus, the existence of an earlier investment in the production of good \( i \) affects the entry threshold for good \( j \) at a later time. This is because the monopolist must take into account the effect of investment in the production of good \( j \) on the already existing cashflow from good \( i \). On the other hand, the existence of the possibility of investing in the production of good \( j \) at a later time does not affect the entry threshold for good \( i \). The intuition for this is that, having the possibility of investing in the production of good \( j \) is equivalent to having the possibility of adding an additional cashflow at a fixed cost \( I_j \).

**Simultaneous Investment**

In this brief section we solve the monopolist’s entry threshold for simultaneous investment in the two goods. It must be pointed out that, as in the sequential investment decision, we are going to examine only the issue of optimal timing of entry conditional on investment being undertaken simultaneously, and not whether simultaneous investment is optimal or not. At the critical profit level for simultaneous investment, the monopolist gives up the option to invest simultaneously in the two goods and incurs the investment cost \( I_A + I_B \). In exchange the monopolist obtains the present discounted value of the cashflows from the combined production of the two goods. Then, by using standard techniques, the value matching condition can be written as

\[
C_3 R_i^* R_i^* K_i^* + I_A + I_B = \frac{\hat{R}_i^*}{W_i}.
\]
where \( C_3 \) is a constant to be determined using the boundary conditions. The smooth pasting condition is

\[
K_i \bar{R} C_3 \bar{R}^{-1} \beta_1^{T} = \bar{R}^{-1} f_1\]

Thus, the entry threshold for simultaneous investment in the two goods is

\[
\bar{h} = \left[ \frac{K_i}{K_i \bar{h} - \bar{h}} \right]^{\frac{1}{\beta_1}} - \left[ W_B I_A + I_B f_1 \right]
\]

Note that, it can be verified from \( \bar{h} \) that \( \bar{h} = \bar{h}^* \) if \( \bar{h} = \bar{h}^* \). The intuition behind this is that, \( \bar{h} = \bar{h}^* \) can be roughly interpreted as the case where the sequential investment in good \( i \) first and good \( j \) later collapses to simultaneous investment.

**Investment Strategies**

The most interesting feature of our analysis is that, despite the normalization proposed in section 3, we find that whenever we account for the possible interaction between the demands of the goods, the investment decision problem is more complex than the textbook case and the results (not only the timing but also the order of investment) might be entirely different.

In order to analyze the monopolist’s investment decision problem when the demand function is given by equation \( \beta_1 \) we classify the different investment strategies that are available to the monopolist into the following projects:

- **(P1)** Enter the production of good \( A \) only;
- **(P2)** Enter the production of good \( B \) only;
- **(P3)** Enter the production of goods \( A \) and \( B \) simultaneously;
- **(P4)** Enter the production of good \( A \) first, and good \( B \) later, without exiting the production of good \( A \);
- **(P5)** Enter the production of good \( B \) first, and good \( A \) later, without exiting the production of good \( A \);
- **(P6)** Enter the production of good \( A \) first, and then exit from its production at a later date upon entry in the production of good \( B \);
- **(P7)** Enter the production of good \( B \) first, and then exit from its production at a later date upon entry in the production of good \( A \);

**Choice of Strategy and Properties of the Goods**

In this section we derive the relationships between the parameter values and the monopolist’s optimal investment decision. It is clear from section 4 that more than one of the investment projects might be feasible with a given range of parameter values. (for example, \( \bar{h} \) or \( \bar{h}^* \) may hold simultaneously, so that, sequential investment in either order is feasible). When more than one project is feasible for a given range of parameter values, we need to find out which of these projects is optimal. This is done by comparing the present discounted values of the feasible projects and choosing the feasible project with the highest present discounted value.

Note that, it is clear from the sequential entry thresholds \( \bar{h} \) that the sequential investment projects \( \bar{h} \) where investment takes place in the production of good \( A \) first and good \( B \) later, are not feasible if \( \bar{h} \) or \( \bar{h}^* \). Analogously, the sequential investment projects \( \bar{h} \) that are not feasible if \( \bar{h} \) and \( \bar{h}^* \) as follows.

If the monopolist has invested in both goods, it is obvious that she will stop producing one of them, say good \( i \), only if

\[
\bar{R} = L_{ij} \beta_1 > \bar{R}_{ij} \]

If a project involving sequential investment in good \( i \) first and good \( j \) later (either with or without exiting the production of good \( i \) is feasible and the value of \( Y \) at the moment when the decision has to be taken is less than \( \bar{Y}_j \), then it can be verified that this project will yield a higher present discounted value than each of the two projects involving investment in good \( i \) only and investment in the two goods simultaneously. Thus, whenever either \( \bar{h} \) or \( \bar{h}^* \) (either \( \bar{h}^* \) or \( \bar{h} \)) is feasible and the value of \( Y \) at the moment when the decision has to be taken is less than \( \bar{Y}_j \), it is suboptimal for the monopolist to choose \( \bar{h} \) or \( \bar{h}^* \).

The monopolist’s problem of choosing an optimal investment strategy will crucially depend on the value of \( Y \) at the moment at which the decision has to be taken and the relationship between the demands for the two goods. Depending on the value of \( \bar{R} \) relative to \( L_{ij} \) and \( L_{ii} \), we classify the relationship between the goods in the usual way. The two goods are \( \beta_1 \) independent if

\[
\bar{R} = L_{ij} + L_{ii} > \bar{R}_{ij} \] complements if \( \bar{R} > L_{ij} + L_{ii} \); and \( \beta_1 \) substitutes if \( \bar{R} < L_{ij} + L_{ii} \).

In the rest of this section, using the entry thresholds derived in section 4 and keeping in mind the observations made above,
we will analyze the monopolist’s investment decision problem when the goods are independent, complements and substitutes in turn.

(1) A and B are independent

When the goods are independent, we know that projects $P_6$ and $P_7$ are ruled out. Our normalization implies that project $P_4$ is feasible and project $P_5$ is not feasible. So if $Y < Y_B^D$ (i.e., $Y < Y_B$), then project $P_4$ dominates projects $P_1$ and $P_3$. Also, it is obvious that project $P_2$ will have a lower present discounted value than project $P_4$ (project $P_3$) if $Y < Y_B^D$ (i.e., $Y > Y_B$).

Therefore, when the goods are independent, the sequential project $P_4$ is optimal if $Y < Y_B^D$, and the simultaneous project $P_3$ becomes optimal by default if $Y \geq Y_B^D$. These results coincide with those obtained by following the textbook approach.

(ii) A and B are complements

When the goods are complement, we know that projects $P_6$ and $P_7$ are ruled out. Also, it can be verified that $\frac{L_B^a}{\bar{Y}_B^L} \leq \frac{L_B^b}{\bar{Y}_B^L}$. So project $P_5$ is not feasible. Then, depending on the degree of complementarity between the two goods, the optimal strategy could be the simultaneous investment project $P_3$ or the sequential investment project $P_4$.

Suppose $\frac{L_B^a}{\bar{Y}_B^L} \leq \frac{L_B^b}{\bar{Y}_B^L}$, which in some sense corresponds to a high degree of complementarity between the two goods. Then project $P_4$ is not feasible. By using the entry thresholds in $P_3$ and $P_7$, it can be also verified that $\bar{Y}_B^L \geq Y_B^D$. Hence, whatever be the value of $Y$ at the moment when the decision has to be taken, the present discounted value of project $P_3$ must be greater than the present discounted value of projects $P_1$ and $P_2$. Thus, given any $Y$, it is optimal to choose project $P_3$ with entry at max $\{Y, \bar{Y}_B^L\}$.

Suppose $\frac{L_B^a}{\bar{Y}_B^L} > \frac{L_B^b}{\bar{Y}_B^L}$, which in some sense corresponds to a lower degree of complementarity between the two goods. Then the sequential investment project $P_4$ is feasible. So we know that project $P_4$ dominates project $P_1$ and $P_3$ if the value of $Y$ at the moment when the decision has to be taken is less than $\bar{Y}_B$. It can also be verified that $Y_B^D = \bar{Y}_A < \bar{Y}_B < Y_B^D$. Thus, it is obvious that projects $P_1$ and $P_2$ cannot be optimal for any value of $Y$. Therefore, given any $Y < Y_B$, it is optimal to choose project $P_4$ with entry in good $A$ at max $\{Y, Y_B^D\}$ and entry in good $B$ at $Y_B$. On the other hand, if $Y \geq Y_B$, then project $P_3$ with simultaneous entry at $Y$ becomes optimal by default.

(iii) A and B are substitutes

When the goods are substitutes, depending on the degree of substitutability, we look at three different cases. The first two are extreme cases where the demand for one good completely wipes out the demand for the other good, that is, the monopolist’s instantaneous profit is maximized by selling only one good even when she has invested in both goods.

Case (i) $R = L_B^D \geq L_B^D$

In this case we can obviously rule out the simultaneous investment project $P_3$ and the sequential investment projects $P_4$ and $P_6$. Also, given our normalization and $L_B^a \geq L_B^b$, it can be verified that project $P_1$ has a higher present discounted value than project $P_2$ for any $Y$, and hence, project $P_2$ is suboptimal.

Suppose $\frac{L_B^a}{\bar{Y}_B^L} \geq \frac{L_B^b}{\bar{Y}_B^L}$ or $L_B^a = L_B^b$. Then it is clear from the entry thresholds in $P_3$ and $P_7$ that the sequential investment project $P_7$ is not feasible. Thus, given any $Y$, it is optimal to undertake project $P_1$ by investing in the production of good $A$ at max $\{Y, Y_B^D\}$.

Suppose $\frac{L_B^a}{\bar{Y}_B^L} < \frac{L_B^b}{\bar{Y}_B^L} < K$ so that project $P_7$ is feasible. Then, given any $Y < \bar{Y}_A$, the monopolist must compare the present discounted values of project $P_1$ and project $P_7$ and choose the project with the higher present discounted value. In the appendix we characterize the conditions under which the present discounted value of $P_1$ is at least as large as the present discounted value of $P_7$.

Clearly, project $P_1$ becomes optimal by default if $Y \geq \bar{Y}_A$.

Case (ii) $R = L_B^D > L_B^D$

As in the previous case, it is obvious that we can rule out projects $P_3$, $P_5$, and $P_7$. Using our normalization and the entry thresholds in $P_3$ and $P_7$, it can be verified that $Y_B^D = Y_A < Y_B < \bar{Y}_B < K$. Thus, the sequential investment project $P_6$ is feasible, which implies that project $P_1$ is suboptimal if $Y < \bar{Y}_B$. Given $L_B^a > L_B^b$, it can be verified that project $P_2$ has a higher present discounted value than project $P_1$ for $Y \geq \bar{Y}_B$. So project $P_1$ remains suboptimal even if $Y \geq \bar{Y}_B$.

Therefore, given any $Y < \bar{Y}_B$, the monopolist must compare the present discounted values of projects $P_2$ and $P_6$ and choose the project with the higher present discounted value. In the appendix we characterize the conditions under which the present discounted value of $P_2$ is at least as large as the present discounted value of $P_6$ for $Y < \bar{Y}_B$. When $Y \geq \bar{Y}_B$, project $P_2$ becomes optimal by default.

Case (iii) $R > \max\{L_B^a, L_B^b\}$

Clearly, projects $P_6$ and $P_7$ can be ruled out in this case. Using our normalization and the entry thresholds in $P_3$ and $P_7$, it can be verified that $Y_B^D = Y_A < Y_B^D < \bar{Y}_B < K$. So the sequential investment project $P_4$ is feasible, which implies that projects $P_1$ and $P_3$ will not be undertaken if $Y < \bar{Y}_B$. Furthermore, it is obvious that project $P_3$ dominates project $P_1$ for $Y \geq \bar{Y}_B$. Thus, project $P_1$ is suboptimal for any $Y$. 
It is possible to have either \( \frac{j_1}{R Y L} \leq \frac{j_2}{R Y L} \) or \( \frac{j_2}{R Y L} < \frac{j_1}{R Y L} \). We will analyze the investment decision problem under each of these two possibilities.

(a) \( \frac{j_1}{R Y L} \leq \frac{j_2}{R Y L} \)

In this case the sequential investment project \( P5 \) is feasible. It can also be verified that project \( P3 \) has a higher present discounted value than project \( P2 \) for all \( Y \). Therefore, the sequential investment project \( P4 \) is optimal for \( Y < \bar{Y}_B \). Needless to say, the simultaneous investment project \( P3 \) becomes optimal by default if \( Y \geq \bar{Y}_B \).

(b) \( \frac{j_2}{R Y L} < \frac{j_1}{R Y L} \)

In this case the sequential investment project \( P5 \) is feasible, which implies that projects \( P2 \) and \( P3 \) will not be undertaken if \( Y < \bar{Y}_A \). It is also obvious that project \( P5 \) dominates project \( P2 \) for \( Y \geq \bar{Y}_A \). So project \( P2 \) is suboptimal for any \( Y \). Therefore, for any \( Y < \max \{ \bar{Y}_A, \bar{Y}_B \} \), it is optimal to undertake either project \( P4 \) or project \( P5 \) whichever has the higher present discounted value. In the appendix we characterize the conditions under which the present discounted value of \( P4 \) at least as large as the present discounted value of \( P5 \) for \( Y < \max \{ \bar{Y}_A, \bar{Y}_B \} \).

When \( Y \geq \max \{ \bar{Y}_A, \bar{Y}_B \} \), it is clear that the simultaneous investment project \( P3 \) becomes optimal by default if \( \bar{Y}_B < \bar{Y}_A \), and project \( P4 \) becomes optimal by default if \( \bar{Y}_A < \bar{Y}_B \).

Summary of Results

We have proposed a framework for studying the optimal investment strategy of a monopolist who faces the problem of choosing products to produce in an interrelated. Some of the key results that we can draw from our analysis are as follows:

1. Under the sequential investment strategies (i.e. projects \( P4 \) and \( P5 \)), the good in which investment takes place first has the same entry threshold as the textbook case. However, the good in which investment takes place later has an entry threshold which is different from the textbook case (except when the goods are independent). The intuition for this result is the following. The monopolist can ignore the interrelationship between the goods when deciding the investment in the first good, because having the possibility of investing in the second good is equivalent to having the opportunity of adding to the existing cashflow at a fixed investment cost. On the other hand, when deciding the investment in the second good, the monopolist must take into account the effect of this investment on the existing cashflow from the first good.

2. It may be optimal for the monopolist to choose a sequential investment strategy in which the order of investment is different from the textbook case. More precisely, although our normalization implies investment in the production of good \( A \) first, and production of good \( B \) later for the textbook case, when the goods are substitutes, it may be optimal for the monopolist to invest in good \( B \) first and good \( A \) later (i.e. projects \( P5 \) for \( P7 \)) under certain conditions which are roughly linked to the degree of substitutability.

3. When the simultaneous investment project \( P3 \) is non-trivially optimal (i.e. for a given \( Y \) such that \( Y \leq \bar{Y} \)), which happens if the goods are complements and the degree of complementarity is sufficiently high for \( \frac{j_1}{R Y L} \leq \frac{j_2}{R Y L} \) to hold, the entry threshold is the same as the textbook case for good \( A \) (in particular, \( \bar{Y} = Y_A \)) while it is different from the textbook case for good \( B \). The intuition for this is that, the benefit from a sufficiently high degree of complementarity may outweigh the benefit from waiting to undertake the production of the second good.

4. The choice of an optimal investment strategy crucially depends on the relationship between the goods and the value of \( Y \) at the moment when the decision has to be taken. In those situations where the value of \( Y \) does not lead to a decision by default, we can draw the following broad conclusions:

4.1. When the goods are independent, the optimal strategy is to invest in good \( A \) first and good \( B \) later, without exiting the production of good \( A \).

4.2. When the goods are complements, the optimal strategy is to invest simultaneously in both goods if the degree of complementarity is sufficiently high; and to invest in good \( A \) first and good \( B \) later, without exiting the production of good \( A \), if the degree of complementarity is not sufficiently high.

4.3. When the goods are extreme substitutes in the sense that the demand for one wipes out the demand for the other, depending on the circumstance, the optimal strategy is either investment in only one good, or sequential investment in which production of the first good is terminated once investment in the second good takes place.

4.4. When the goods are substitutes but not extreme substitutes, depending on the circumstance, the optimal strategy is one of the two sequential investment strategies in which production of the first good continues even after investment in the second good.
REFERENCES


Appendix

Given any \( Y \), for each \( t = 1, \ldots, 7 \), we denote the present discounted value of project \( P \mid f_i \) by \( P \mid y_i \) and we also introduce the notation

\[
K \mid y_i = \left\{ \begin{array}{ll} K_i & \text{if } 1 \leq t < K_i \\
K_i & \text{if } t = K_i \end{array} \right.
\]

(III) \( A \) and \( B \) are substitutes:

Case (i) \( R = L_D^B > L_D^A \).

In this case we need to characterize conditions under which \( P_1 \mid y_i \geq 0 \) for \( Y < \#_A \) when \( \frac{L_D^B}{K_i} \leq \frac{L_D^A}{K_i} < K \) (i.e.
\( Y_{1D} > \#_A < K \)). We consider three different possibilities: (i.1) \( Y < Y_{1D} \); (i.2) \( Y_{1D} \leq Y < Y_{2D} \); (i.3) \( Y_{2D} \leq Y < \#_A \).

(i.1) Suppose \( Y < Y_{1D} \). Then we have

\[
P_1 \mid y_i = \left( Y \right)^{K_i} \left[ \frac{L_D^B}{K_i} \right] \left[ \frac{L_D^A}{K_i} \right] I_A \]

Thus, it can be verified that \( P_1 \mid y_i \geq 0 \) if and only if

\[
\left( L_D^A, L_D^B \right) \geq \left( Y \right)^{K_i} \left[ \frac{L_D^B}{K_i} \right] I_A
\]

(i.2) Suppose \( Y_{1D} \leq Y < Y_{2D} \). Then the expression for \( P_1 \mid y_i \) remains the same as in (i.1) and

\[
P_1 \mid y_i = \left[ \frac{L_D^B}{K_i} \right] I_A
\]

Thus, it can be checked that \( P_1 \mid y_i \geq 0 \) if and only if

\[
\left( L_D^B, L_D^A \right) \geq \left[ \frac{L_D^B}{K_i} \right] I_A
\]

(i.3) Suppose \( Y_{2D} \leq Y < \#_A \). Then the expression for \( P_1 \mid y_i \) remains the same as in (i.2) and

\[
P_1 \mid y_i = \left[ \frac{L_D^B}{K_i} \right] I_A
\]

Thus, it can be verified that \( P_1 \mid y_i \geq 0 \) if and only if

\[
\left( L_D^B, L_D^A \right) \geq \left[ \frac{L_D^B}{K_i} \right] I_A
\]

Case (ii) \( R = L_D^B > L_D^A \).

In this case \( Y_{1D} = \#_A < Y_{2D} < \#_A < K \) holds. We need to characterize conditions under which \( P_2 \mid y_i \geq 0 \) for \( Y < \#_A \).

By arguments analogous to those used in (i), we can conclude the following:
Hence, it can be verified that $Y \leq Y_D^\circ$, then $P_{\leftrightarrow} Y\uparrow_i? P_{\leftrightarrow} Y\downarrow_i \geq 0$ if and only if
\[
\left[ \left( L_D^\circ \uparrow + I_A \right) \right] \left( I_B^\circ \uparrow + I_A \right) \geq 0.
\]

(ii.2) If $Y_D^\circ \leq Y < Y_D^\circ$, then $P_{\leftrightarrow} Y\uparrow_i? P_{\leftrightarrow} Y\downarrow_i \geq 0$ if and only if
\[
\left[ K \cdot Y \uparrow - \left( \frac{Y}{Y_D^\circ} \right) \right] \left( I_B^\circ \uparrow + I_A \right) \geq 0.
\]

(ii.3) If $Y_D^\circ \leq Y < Y_D^\circ$, then $P_{\leftrightarrow} Y\uparrow_i? P_{\leftrightarrow} Y\downarrow_i \geq 0$ if and only if
\[
\left[ \left( L_D^\circ \uparrow + I_A \right) \right] \left( I_B^\circ \uparrow + I_A \right) \geq 0.
\]

Case (iii) $R > \max \cdot L_D^\circ, L_D^\circ, L_D^\circ, L_D^\circ, L_D^\circ,
\]

In this case we need to characterize conditions under which $P_{\leftrightarrow} Y\uparrow_i? P_{\leftrightarrow} Y\downarrow_i \geq 0$ for $Y < \max \cdot L_D^\circ, L_D^\circ, L_D^\circ, L_D^\circ, L_D^\circ,
\]

(iii.b.1) $Y < Y_D^\circ$,

(iii.b.2) $Y_D^\circ \leq Y < Y_D^\circ$,

(iii.b.3) $Y_D^\circ \leq Y < \min \cdot \#_A, \#_A$.

(iii.b.1) Suppose $Y < Y_D^\circ$. Then, for $i, j = A, B, i \in E_i$, we have
\[
P_{ij} Y\uparrow_i \downarrow_j = \left( \frac{Y}{Y_D^\circ} \right) K \cdot \left[ \left( L_D^\circ \uparrow + I_A \right) \right] K \cdot \left( I_B^\circ \uparrow + I_A \right)
\]

where $P_{ij} Y\uparrow_i = P_{ij} Y\downarrow_i$ and $P_{ij} Y\uparrow_j = P_{ij} Y\downarrow_i$. Thus, $P_{ij} Y\uparrow_i \downarrow_j \geq 0$ if and only if
\[
\left[ \left( L_D^\circ \uparrow + I_A \right) \right] \left( I_B^\circ \uparrow + I_A \right) \geq 0.
\]

(iii.b.2) Suppose $Y_D^\circ \leq Y < Y_D^\circ$. Then the expression for $P_{ij} Y\uparrow_i \downarrow_j$ remains the same as in (iii.b.1) and we have
\[
P_{ij} Y\uparrow_i \downarrow_j = \left( \frac{Y}{Y_D^\circ} \right) K \cdot \left( I_B^\circ \uparrow + I_A \right)
\]

Hence, it can be verified that $P_{ij} Y\uparrow_i \downarrow_j \geq 0$ if and only if
\[
\left[ \left( L_D^\circ \uparrow + I_A \right) \right] \left( I_B^\circ \uparrow + I_A \right) \geq 0.
\]

(iii.b.3) Suppose $Y_D^\circ \leq Y < \min \cdot \#_A, \#_A$. Then the expression for $P_{ij} Y\uparrow_i \downarrow_j$ remains the same as in (iii.b.2) and we have
\[
P_{ij} Y\uparrow_i \downarrow_j = \left( \frac{Y}{Y_D^\circ} \right) K \cdot \left( I_B^\circ \uparrow + I_A \right)
\]

Hence, it can be verified that $P_{ij} Y\uparrow_i \downarrow_j \geq 0$ if and only if
\[ \mathcal{L}_A \cap \mathcal{L}_B \cap \mathcal{B} \cap \mathcal{M} \cap \mathcal{N} \cap \mathcal{O} \cap \mathcal{P} \cap \mathcal{Q} \cap \mathcal{R} \cap \mathcal{S} \cap \mathcal{T} \cap \mathcal{U} \cap \mathcal{V} \cap \mathcal{W} \cap \mathcal{X} \cap \mathcal{Y} \cap \mathcal{Z} \]

\[ \mathcal{K} \mathcal{Y} \mathcal{F}( \mathcal{L}_A \cap \mathcal{L}_B \cap \mathcal{B} \cap \mathcal{M} \cap \mathcal{N} \cap \mathcal{O} \cap \mathcal{P} \cap \mathcal{Q} \cap \mathcal{R} \cap \mathcal{S} \cap \mathcal{T} \cap \mathcal{U} \cap \mathcal{V} \cap \mathcal{W} \cap \mathcal{X} \cap \mathcal{Y} \cap \mathcal{Z} ) \]

\[ \text{for } \mathcal{R} \cap \mathcal{L}_A \cap \mathcal{L}_B \cap \mathcal{B} \cap \mathcal{M} \cap \mathcal{N} \cap \mathcal{O} \cap \mathcal{P} \cap \mathcal{Q} \cap \mathcal{R} \cap \mathcal{S} \cap \mathcal{T} \cap \mathcal{U} \cap \mathcal{V} \cap \mathcal{W} \cap \mathcal{X} \cap \mathcal{Y} \cap \mathcal{Z} \geq 0. \]