THE OPTIMAL TIMING OF THE INTRODUCTION OF NEW PRODUCTS[1]

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Abstract

This paper addresses the effects for partial equilibrium models of relaxing one of the critical underlying assumptions of the textbook approach (Dixit and Pyndick, 1994) to investment under uncertainty: either the potential investor has access to a single project or she can consider competing (or complementary) projects independently. This paper studies the investment decision of a multi-product monopolist where the projects exhibit interdependence between the cash flows of different products. We derive the optimal entry time for each product and show that both the choice and timing of investment is different from that suggested by the textbook approach. The decision to produce related goods simultaneously or sequentially crucially depends on their degree of substitutability or complementarity.

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1 Introduction

For a long time now, economists have acknowledged the power of the options approach in the modeling of investment decisions. According to this framework, such decisions should be modeled so as to take into account the irreversibility of the investment and the uncertainty surrounding them. Thus, as in the theory of options in financial markets, there exists an option value of waiting for better information, which is taken into account when deciding the timing of investment. This approach has been embraced by those modeling phenomena which can be cast under the optimal stopping problem. Regarding investment decisions, these models tend to deal with investors who have access to a single project or consider (complementary) competing projects independently. However, most investment decisions involve the choice amongst several interdependent projects and not amongst independent ones. Say, for example, the introduction of new financial instruments or a university considering in opening a new master program, complementing the undergraduate programs. Introduction of more advanced versions of existing software by the same company also falls into this category. Therefore, the literature has remained silent as to how to deal with this kind of decisions from the options approach. Our paper intends to tackle this situation. In a stylized framework, we consider the effects of the presence of (complementary) competing projects over the timing of investment.

To capture this idea, we will focus on the optimal entry decision of a multi-product monopolist. Within this framework, we would like to pose the following question: if faced with two different (complementary/competing/independent) products, how does the interdependence between them affect her decision of investing in either one or both of them and the timing of those decisions? Our contribution is to develop and study a methodology that provides an answer to this question. Our methodology allows us to value complementary (competing) risky projects and, therefore, we can determine the optimal timing of investment decisions within this framework.

This question is relevant in its own right. Even though this paper will focus on the multi - product monopolist, the problem we deal with pertains a broader class of circumstances. As aforementioned, most investment decisions involve interrelated, and not independent, projects. Our methodology provides insight as to how the cash flow interactions amongst different investment projects determine the optimal investment strategy. This can prove to be an important tool in order to be able to determine the composition of a portfolio at a micro level and the implications for the economy's portfolio. Moreover, further insight into that composition and the interdependences between the projects can aid us in our predictions about how the economy's portfolio of investment projects may respond to alterations in cash flows, say, as a consequence of a shock to the economy.

What is more, product segmentation analysis within our framework should be of interest to the Industrial Organization economists. There is a vast literature on the topic. However, it deals with the motives for sequential introduction of products in a different way we do. Moorthy and Png (1982) [12] consider the effects of introducing different quality levels of a durable good in the presence of two market segments with different valuations for the good. Norton and Bass (1987) [13] and Levinthal and Purohit (1989) [9] model sequential introduction as a consequence of technological development. Other models, such as Wilson and Norton (1987) [17], analyze the influence diffusion effects have on the timing of the introduction of product line extensions. Notice, then, that in this literature sequential introduction arises as a way to avoid the cannibalism between substitute products, such as in [12], or the diffusion effects on demand. However, when choosing amongst the different strategies, the monopolist faces no uncertainty whatsoever. Our contribution to this literature is, then, threefold. First, we incorporate an additional motive for sequential introduction of products, besides the nature of the relationship between the two goods: the value of waiting for better information when the monopolist is faced with projects with uncertain returns. This a departure from [12] where introduction dates are fixed exogenously, and from [17] where the optimal timing depends on diffusion and not on the uncertainty surrounding the returns of the project. A relevant question is how the presence of uncertainty affects the known results on cannibalism for the substitute products case. Second, we analyze the possibility of complementary products, something that has not been studied in the literature. The third contribution is technical. By using the tools of the options approach, we hope that they will be found useful for studying other dynamic problems in the literature.

Finally, our paper can also be considered a contribution to the optimal investment literature. The first paper to address interdependent investment project is Reiter's [15]. The author considers the possibility of several projects whose cash flows are pairwise interdependent, and develops an algorithm for choosing amongst the possible combinations. However, the paper does not consider investment as an optimal stopping problem and, thus the value of waiting is not taken into account. What is more, investment in several projects is done in a simultaneous fashion, whereas we allow the monopolist to choose the timing of different investments optimally. Our paper is an important contribution to this literature, both conceptually and methodologically. Another paper that deals with interdependent projects is Marglin. The author adds a time dimension to the problem. However, the interdependence is through the budget constraint and not through output - pricing decisions.

Sreedharan and Wein [16] consider a model for an n - stage multiproduct investment program in an environment with stochastic demands. They solve their model using dynamic programming techniques, where the optimal schedule is determined as that which minimizes total expected costs. The latter definition of optimal is in contrast with the one we consider in the paper. What is more, they analyze a problem where the project is fixed and one can only choose the phasing.

Erlenkotter and Trippi [7] develop a model for simultaneously planning expansion, output and pricing decisions over time for a firm in a growing market. However, their model allows for the interpretation of investment in interdependent projects. They solve the model in a deterministic setting, thus, no option value of waiting for better information is considered: all the relevant information is known from the start and the firm only has to choose amongst all the possible timing combinations. There is a long list of papers[1] that deal with the timing of capacity expansions, but none of them deal with the possibility of entry and exit from substitute products' markets.

Before providing a brief overview of our main results, a comment is in order. There are two type of distinct investment decisions one would like to model. First, the decision of producing a new good when another one is already under production. This decision will typically affect the profits obtained from the existing good. Then, conditional on having first invested in the production of one good, the optimal decision on producing an additional good will crucially be affected by the nature of the goods that are being produced. Second, we should take into account the ex ante decision of whether to take on the production of the first good when the possibility of undertaking a (complementary) competing project is available. Our framework allows us to model both decisions.

As aforementioned, these decisions appear in a wide variety of investing scenarios. Nevertheless, we will apply these general ideas to the case of a multi - product monopolist who faces stochastic prices in the markets she is considering entry. The investment decision implies a fixed and fully irreversible investment cost. Demand is modeled so as to capture in a most general framework the cases of independence, complementarity and substitutability

[1] See, for example, Erlenkotter [5] and Erlenkotter and Rogers [6]

between the two products. We do away with variable costs for the sake of simplicity.

Regarding the results of our model, when we consider the decision problem ex-ante, we find that not only the standard rule for optimal investment is in general not valid, but also that the sequence of investing decisions might be different from the textbook rule once we allow for the interrelation in the cash flow of the projects.

More particularly, we find that the optimal entry thresholds depend crucially on the degree of substitutability or complementarity between the two goods. When goods are substitutes the investment strategy is most of the times sequential and the entry threshold for the second good cannot be derived using the textbook approach, that is, without taking into consideration the parameters linking the demand functions of the two goods. When goods are complements, the investment strategy is typically simultaneous, and the entry thresholds for both goods differ from that derived using the textbook approach.

We also find that, when goods are substitutes, it might be optimal for the monopolist to invest in one good, say good A, then exit the production of good A (even though there is no option value to exit) and to enter the production of good B.This extends the cannibalism results of **[12]** to our setting: if the goods are substitutes, producing B might eliminate the demand for good A. Thus, the cannibalism phenomena is not exclusive of the adverse selection present in a durable goods market: it can arise in a market with non - durable goods where the source of friction is the uncertainty of the return of the project. Again, notice that in [12] returns are not uncertain: once the monopolist chooses the optimal menu (or single) qualities (quality) she will offer, she knows for certain what her profits will be. Cannibalism in our setting presents itself in the following fashion: if there is an optimal moment to enter B, this will imply that, before that point in time is reached, it might be profitable to enter A. If this happens, when the monopolist enters B, there is no option value of exiting A since this good has no demand. This provides further intuition to the results of **[12]**, extending them to a more general framework than that with two market segments and qualities.

The rest of the paper is organized as follows. The isolated investment projects' case is illustrated in Section 3. Section 4 deals with the sequential investment strategy while Section 5 analyzes the simultaneous investment one. The choice of investment strategy and our main results are discussed in Section 6.Section 7 concludes and points out to further research.

2 The Model

Consider a monopolist who has the possibility of investing in the production of goods A and B. The inverse demand function for good $i = A, B$ is given by

$$P_i = YD_i(Q_i, Q_j), \qquad j = A, B,\ j \neq i, \tag{1}$$

where $D_i(.,.)$ is differentiable. Y is a multiplicative shock which follows a geometric Brownian motion

$$dY = \mu Y dt + \sigma Y dw, \tag{2}$$

where dw is the increment of a Wiener process with $E[dw] = 0$ and $E[(dw)^2] = dt$.[2] Notice that for this demand function, the (instantaneous) quantity produced is deterministic while the price is stochastic.[3] In the inverse demand function, a positive (negative) $\frac{\partial D_i}{\partial Q_j}$, $i \neq j$, implies that the two products are complements (substitutes), and $\frac{\partial D_i}{\partial Q_j} = 0$, $i \neq j$, implies that the two goods are independent of each other. We also assume that the variable costs of production are zero for both goods, and investing in good i requires a fixed and fully irreversible investment cost $I_i > 0$.

3 Independent Products

In this section we consider the case of independent products where the profit stream of each product is independent of the other product. In our framework this corresponds to the case of looking at the investment decision problem of each good by assuming that the quantity of the other is fixed at zero. Thus, we examine the investment decision problem of each good i by setting $Q_j = 0$, $j \neq i$, that is, we use $P_i = YD_i(Q_i, 0)$ as the inverse demand function of good i.

Let $\gamma_i = \max\limits_{Q_i} D_i(Q_i, 0)Q_i$ for each $i = A, B$. Then the instantaneous cashflow from good i is $\gamma_i Y_t$. So the value of the project of investing in good i as a function $V_i(Y)$ of the

[2] In solving the optimal entry for each product, we assume that the investor is a monopolist, thereby, abstracting away from oligopolistic interaction. The introduction of a new product within a strategic framework may serve the purpose of preempting the entry of new competitors and this may partially destroy the option value of waiting (see Lambrecht and Perraudin (1997)).

[3] The effect of a multiplicative shock is to change the price intercept of the demand function leaving unchanged the quantity intercept. This could, for example, be interpreted as a "fashion" shock.

shock Y (which is the expected discounted stream of cashflows) is given by

$$V_i(Y_t) = \int_t^{\infty} \gamma_i Y_s e^{-\rho(s-t)} ds = \frac{\gamma_i Y_t}{\rho - \mu} \tag{3}$$

The problem of determining the point at which it is optimal to invest in good i is an optimal stopping problem. To analyze this problem, let us denote the value of the option to invest in good i (i.e. the value of the investment opportunity, which is the right but not the obligation to undertake the investment) as a function of the shock Y by $F_i(Y)$. Then the Bellman equation of the optimal stopping problem is

$$\rho F_i(Y) dt = E[dF_i(Y)]$$

where ρ is the monopolist's discount factor. Using Ito's Lemma, the Bellman equation becomes

$$(1/2)\sigma^2 Y^2 F_i''(Y) + \mu Y F_i'(Y) - \rho F_i(Y) = 0 \tag{4}$$

The option value $F_i(Y)$ must also satisfy the following usual boundary conditions:

$$F_i(0) = 0 \tag{5}$$
$$F_i(Y_i^*) = V_i(Y_i^*) - I_i \tag{6}$$
$$F_i'(Y_i^*) = V_i'(Y_i^*) \tag{7}$$

where Y_i^* is the critical value of the shock Y at which it is optimal to invest in good i. Condition (5) is an implication of the stochastic process in (2). Conditions (6) and (7) are respectively the value-matching and smooth-pasting conditions.

The general solution of the differential equation (4) can be written as

$$F_i(Y) = C_1 Y^{\beta_1} + C_2 Y^{\beta_2}$$

where C_1 and C_2 are constants to be determined, and β_1 and β_2 are the roots of the characteristic equation

$$(1/2)\sigma^2 \beta(\beta - 1) + \mu\beta - \rho = 0$$

It can be verified that

$$\beta_1 = (1/2) - (\mu/\sigma^2) + \sqrt{((\mu/\sigma^2) - (1/2))^2 + (2\rho/\sigma^2)} \quad > \quad 1$$
$$\beta_2 = (1/2) - (\mu/\sigma^2) - \sqrt{((\mu/\sigma^2) - (1/2))^2 + (2\rho/\sigma^2)} \quad < \quad 0$$

So the boundary condition (5) implies that $C_2 = 0$ and $F_i(Y) = C_1 Y^{\beta_1}$. By substituting this into the boundary conditions (6) and (7) and using (3), we find that the critical value Y_i^* at which it is optimal to invest in good i is given by

$$Y_i^* = \left[\frac{\beta_1}{\gamma_i(\beta_1 - 1)}\right](\rho - \mu)I_i \tag{8}$$

For the critical value in (8), we can show that the expected time of investment in good i is

$$E(T_i^*) = \left[\frac{\ln Y_i^*}{\mu - (\sigma^2/2)}\right] = \left[\frac{\ln(\beta_1(\rho - \mu)I_i/\gamma_i(\beta_1 - 1))}{\mu - (\sigma^2/2)}\right] \tag{9}$$

The investment rule given by (8) or (9) has the interpretation that the investment is postponed, the larger is the variance of the stochastic process, the smaller is the drift parameter, the larger is the investment cost, the larger is the monopolist's discount factor, or the smaller is the demand for the product.

4 Sequential investment

In this section we consider the problem of determining the critical points at which it is optimal to invest when the monopolist's project involves sequential investment in the two goods, with investment in good i $(= A, B)$ in stage one and investment in good j $(= A, B;\ j \neq i)$ in stage two. We will solve the investment problem in two steps by working backwards. In the first step, we will workout the value of the option to invest in good j in stage two and use this to find the critical value of Y at which it is optimal to invest in good j. In the second step, we will workout the value of the option to invest in good i in stage one and use this to find the critical value of Y at which it is optimal to invest in good i.

4.1 Stage Two

Let $\eta = \max_{(Q_A,Q_B)} [D_A(Q_A,Q_B)Q_A + D_B(Q_A,Q_B)Q_B]$. Then, the instantaneous cashflow from the combined production of goods A and B is ηY_t. Therefore, the value of the finished project on completion of investment in good j (which is the expected discounted stream of cashflows from combined production) as a function of the shock is given by:

$$V_{ij}(Y_t) = \int_t^{\infty} \eta Y_s e^{-\rho(s-t)} ds = \frac{\eta Y_t}{\rho - \mu} \tag{10}$$

To analyze the optimal stopping problem in stage two, let us denote the value of the option to invest in good j, given that the monopolist has already invested in good i, as a function of the shock, by $F_{j2}(Y)$.

The Bellman equation of the stage two problem is:

$$\rho F_{j2}(Y)\, dt = \gamma_i Y dt + E_t [dF_{j2}(Y)] \tag{11}$$

The intuition behind the last expression is straightforward. At any moment t, the monopolist has either of two options. On the one hand, she can exercise the option (a strategy which has cost $F_{j2}(Y)$). On the other hand, she can wait, earn the cash flow from project i, in which she has already invested, together with the change in value of the option (a strategy which has a marginal expected benefit of $\gamma_i Y dt + E_t [dF_{j2}(Y)]$). It must be the case that, at the optimal timing of investing in good j, the monopolist is indifferent between both.

It follows from applying Ito's Lemma to equation (11) that it reduces to:

$$\rho F_{j2}(Y) - \mu Y F_{j2}^{'}(Y) - \frac{1}{2}\sigma^2 Y^2 F_{j2}^{''}(Y) = \gamma_i Y \tag{12}$$

The general solution to this differential equation is given by:

$$F_{j2}(Y) = K_1 Y^{\beta_1} + K_2 Y^{\beta_2} + K_3 Y \tag{13}$$

where β_1 and β_2 are the same as before[4], and K_1, K_2 and K_3 are constants to be

[4] The reader may check it using standard differential equations techniques in equation 12.

determined from the boundary, value matching and smooth pasting conditions:

$$F_{j2}(0) = 0 \tag{14}$$

$$F_{j2}\left(Y_{j2}^{*}\right) = V_{ij}\left(Y_{j2}^{*}\right) - I_j \tag{15}$$

$$F_{j2}^{'}\left(Y_{j2}^{*}\right) = V_{ij}^{'}\left(Y_{j2}^{*}\right) \tag{16}$$

Equation (14) follows from the stochastic process considered for Y. Equations (15) and (16) determine optimality conditions that must hold when deciding to invest.

The next lemma describes the functional form of the option value:

Lemma 1

$$F_{j2}(Y) = K_1 Y^{\beta_1} + \frac{\gamma_i Y}{\rho - \mu} \tag{17}$$

where K_1 is a real number determined by the parameters in equations (14), (15) and (16)

Proof. It follows from the Bellman equations and the aforementioned conditions ■

Notice the similarities and differences with the option value in the textbook case. The first term in equation (17) is the analogue of the one we had found in the independent goods case, up to a change in units. The second term is completely new: it is a consequence of the presence of the investment in good i. The value of the option of postponing the decision to invest in good j, given that the monopolist has already invested in good i, must take into account the fact that not investing in good j gives the possibility of earning the cash flow stream generated by good i. That is what the second term stands for.

What is more, the presence of good i will, in general, alter the optimal timing of investment in good j as it is made clear in the following lemma:

Lemma 2 *Let F_{j2} be as in Lemma 1. Then, the value of Y_{j2} which solves equations (15) and (16) is given by:*

$$Y_{j2}^{*} = \left[\frac{\beta_1}{(\eta - \gamma_i)(\beta_1 - 1)}\right](\rho - \mu) I_j \tag{18}$$

and the optimal (expected) time of investment is:

$$E(T_{j2}^*) = \left[\frac{\ln Y_{j2}^*}{\mu - (\sigma^2/2)}\right] = \left[\frac{\ln\left(\beta_1(\rho-\mu)I_j/(\eta-\gamma_i)(\beta_1-1)\right)}{\mu - (\sigma^2/2)}\right] \tag{19}$$

Now that we have the formula for the value of the option F_{j2} we can proceed to analyze stage one.

4.2 Stage One

Suppose the monopolist has not yet invested in good i. She faces two options. She can either invest in good i, which will give her the option to invest in good j or she can wait another period to exercise her option to invest in good i.

Let $F_{i1}(Y)$ be the value of the option to invest in good i, when no other project has been undertaken yet as a function of the shock Y. The Bellman equation for the stage one problem resembles the one we saw in the textbook case:

$$\rho F_{i1}(Y) = E_t\left[dF_{i1}(Y)\right] \tag{20}$$

It follows from Ito's Lemma (applied to equation (20)) that F_{i1} is such that:

$$\frac{1}{2}\sigma^2 Y^2 F_{i1}^{''}(Y) + \mu Y F_{i1}^{'}(Y) - \rho F_{i1}(Y) = 0 \tag{21}$$

Moreover, $F_{i1}(.)$ must satisfy the following boundary, value matching and smooth pasting conditions:

$$F_{i1}(0) = 0 \tag{22}$$

$$F_{i1}(Y_{i1}^*) = F_{j2}(Y_{i1}^*) - I_i \tag{23}$$

$$F_{i1}^{'}(Y_{i1}^*) = F_{j2}^{'}(Y_{i1}^*) \tag{24}$$

Notice that, unlike the textbook case, the value matching and smooth pasting conditions do not include $V_i(Y)$ explicitly: it is already taken into account in the value of the option to invest in good j.

The following lemma characterizes the threshold Y_{i1}^* and the optimal (expected) time of investment in good i in stage one.

Lemma 3 *Let F_{i1} be such that (21) and (22) hold, and let F_{j2} be as in Lemma 1. Then, the value of Y that satisfies equations (23) and (24) is given by:*

$$Y_{i1}^* = \left[\frac{\beta_1}{\gamma_i(\beta_1 - 1)}\right](\rho - \mu)I_i \tag{25}$$

Moreover, the optimal (expected) time of entry in good i is:

$$E(T_{i1}^*) = \left[\frac{\ln Y_{i1}^*}{\mu - (\sigma^2/2)}\right] = \left[\frac{\ln\left(\beta_1(\rho - \mu)I_i/\gamma_i(\beta_1 - 1)\right)}{\mu - (\sigma^2/2)}\right] \tag{26}$$

which is the same as in the independent good case.

Note that the critical point Y_{i1}^* for investment in good i in stage one is exactly the same as the critical point Y_i^* for investment in good i in the independent products case. Therefore, the existence of the possibility of investment in good j in stage two does not affect the critical point of investment in good i. This is intuitive: having the possibility of investment in good j in stage two is equivalent to having the possibility of adding an additional cashflow at a fixed cost I_j.

By comparing equations (8) and (18), it is clear that the critical point Y_{j2}^* for investment in good j in stage two is the same as the critical point Y_j^* for investment in good j in the independent products case if and only if $\eta - \gamma_i = \gamma_j$. However, $\eta - \gamma_i = \gamma_j$ if and only if the products are independent. Thus, unless the two goods are independent, the existence of an earlier investment in good i affects the critical point of investment in good j in stage two. The monopolist must take into account the effect of investment in good j on the already existing cashflow from good i.

5 Simultaneous investment

We now proceed to determine the optimal timing for simultaneous investment in the two goods. Recall from Section 4 that the instantaneous cashflow from the combined production of goods A and B is ηY_t. Thus, the function $V_{ij}(Y)$, given in (10), is the value of the project of simultaneous investment. Let $F_{ij}(Y)$ denote the value of the option to invest simultaneously as a function of the shock Y . The corresponding Bellman equation is:

$$\rho F_{ij}(Y)dt = E[dF_{ij}(Y)] \tag{27}$$

The value of the option $F_{ij}(Y)$ must also satisfy the following boundary conditions:

$$\begin{aligned} F_{ij}(0) &= 0 && (28) \\ F_{ij}(Y_s^*) &= V_{ij}(Y_s^*) - I_i - I_j && (29) \\ F'_{ij}(Y_s^*) &= V'_{ij}(Y_s^*) && (30) \end{aligned}$$

where Y_s^* is the critical value of Y at which it is optimal to invest simultaneously in the two goods. Again, condition (28) follows from (2), and conditions (29) and (30) are the value-matching and smooth-pasting conditions, respectively.

By going through similar steps as before, it can be verified that the critical point Y_s^* at which it is optimal to invest simultaneously in the two goods is:

$$Y_s^* = \left[\frac{\beta_1}{\eta(\beta_1 - 1)}\right](\rho - \mu)(I_i + I_j) \tag{31}$$

and the corresponding expected time of simultaneous investment is:

$$E(T_s^*) = \left[\frac{\ln Y_s^*}{\mu - (\sigma^2/2)}\right] = \left[\frac{\ln\left(\beta_1(\rho - \mu)(I_i + I_j)/\eta(\beta_1 - 1)\right)}{\mu - (\sigma^2/2)}\right] \tag{32}$$

It is worth pointing out that, in the case of sequential investment, whenever $Y_{i1}^* = Y_{j2}^*$, the two stages collapse into a single stage and sequential investment reduces to simultaneous investment. Thus, whenever $Y_{i1}^* = Y_{j2}^*$, it is not surprising to observe that $Y_{i1}^* = Y_{j2}^* = Y_s^*$, that is, whenever the sequential investment case reduces to simultaneous investment, the critical value of Y is automatically given by (31). Furthermore, it can be verified that $Y_{i1}^* < Y_s^* < Y_{j2}^*$ whenever $Y_{i1}^* < Y_{j2}^*$.

6 Choice of Investment Strategy

Notice that, when the goods are independent (i.e. $\eta = \gamma_A + \gamma_B$), our analysis in Section 3 shows that the monopolist's optimal strategy is to invest in good i $(= A, B)$ once $Y \geq Y_i^*$. In order to analyze the monopolist's investment decision problem when the goods are not independent (i.e. $\eta \neq \gamma_A + \gamma_B$), we classify the different investment strategies that are available to the monopolist into the following projects:

(P_s) Invest in goods A and B simultaneously;

(P_{ij}) For each $i, j = A, B$ and $i \neq j$, invest sequentially, with investment in good i in stage one and investment in good j in stage two.

We allow for the possibility that $\eta = \gamma_A$ or γ_B. Thus, if $\eta = \gamma_i$, the above mentioned three projects automatically reduce to one of two possibilities: either invest in good i only, or invest in good j ($\neq i$) in stage one and invest in good i in stage two and stop the production of good j.

Given $Y_{i1}^* \leq Y_{j2}^*$ for $i \neq j$ and any Y, let $PV_s(Y)$ and $PV_{ij}(Y)$ be the present discounted values of the simultaneous project (P_s) and the sequential project (P_{ij}) respectively. Then it can be verified that

$$PV_s(Y) = \begin{cases} \left(\frac{Y}{Y_s^*}\right)^{\beta_1} \left[\frac{\eta Y_s^*}{\rho-\mu} - I_i - I_j\right] & \text{if } Y \leq Y_s^* \\ \frac{\eta Y}{\rho-\mu} - I_i - I_j & \text{otherwise} \end{cases} \tag{33}$$

$$PV_{ij}(Y) = \begin{cases} \left(\frac{Y}{Y_{i1}^*}\right)^{\beta_1} \left[\frac{\gamma_i Y_{i1}^*}{\rho-\mu} - I_i\right] + \left(\frac{Y}{Y_{j2}^*}\right)^{\beta_1} \left[\frac{(\eta-\gamma_i) Y_{j2}^*}{\rho-\mu} - I_j\right] & \text{if } Y < Y_{i1}^* \\ \frac{\gamma_i Y}{\rho-\mu} - I_i + \left(\frac{Y}{Y_{j2}^*}\right)^{\beta_1} \left[\frac{(\eta-\gamma_i) Y_{j2}^*}{\rho-\mu} - I_j\right] & \text{if } Y_{i1}^* \leq Y < Y_{j2}^* \\ \frac{\eta Y}{\rho-\mu} - I_i - I_j & \text{otherwise} \end{cases} \tag{34}$$

Clearly, $PV_s(Y) = PV_{ij}(Y)$ for all $Y \geq Y_{j2}^*$. Furthermore, it can be verified that both $PV_s(Y)$ and $PV_{ij}(Y)$ are differentiable, and $PV_s(Y) = PV_{ij}(Y)$ for all Y if $Y_{i1}^* = Y_{j2}^*$.

As mentioned earlier, whenever $Y_{i1}^* < Y_{j2}^*$ for $i \neq j$, we know that $Y_{i1}^* < Y_s^* < Y_{j2}^*$. In this case the following proposition shows that the sequential project (P_{ij}) has a higher present discounted value than the simultaneous project (P_s) at any $Y < Y_{j2}^*$.

Proposition 4 *If $Y_{i1}^* < Y_{j2}^*$ for $i \neq j$, then $PV_{ij}(Y) > PV_s(Y)$ at every $Y < Y_{j2}^*$.*

Proof. See the appendix. ■

The monopolist's problem of choosing an optimal investment strategy and when to invest in which good will crucially depend on the value of Y at the moment when the decision has to be taken. If $Y \geq Y_{j2}^*$, it is clear that the corresponding sequential project

(P_{ij}), $i \neq j$, cannot be optimal. So the simultaneous project (P_s) is optimal whenever $Y \geq \max\{Y^*_{A2}, Y^*_{B2}\}$. Given Y, if (P_{ij}) is the optimal strategy, the monopolist should invest in good i at $\max\{Y, Y^*_{i1}\}$ followed by investment in good j at Y^*_{j2} (where we allow for the possibility of $Y^*_{j2} = \infty$, i.e. $\gamma_i = \eta$, in which case the subsequent investment in good j never takes place). Similarly, given Y, if (P_s) is optimal, the monopolist should invest simultaneously in the two goods at $\max\{Y, Y^*_s\}$.

The monopolist's problem of picking an optimal investment project will also depend on whether the two goods are complements ($\eta > \gamma_A + \gamma_B$) or substitutes ($\eta < \gamma_A + \gamma_B$). We consider these possibilities below.

6.1 Goods A and B are complements

Intuition suggests that when the goods are complements, the optimal strategy for the monopolist is that of simultaneous investment. That is, instead of producing right shoes only, the monopolist should find it optimal to produce both left and right ones. Nonetheless, this result depends on the degree of complementarity between the goods and the relationship between investment costs and the cash flows. The main results for the case of complement goods are summarized in the following propositions:

Proposition 5 *Suppose $\eta > \gamma_A + \gamma_B$ and $\frac{I_A}{\gamma_A} = \frac{I_B}{\gamma_B}$. Then, given any Y, (P_s) is the optimal strategy and $Y^*_s < Y^*_i$ for $i = A, B$.*

Proof. It follows from checking that $Y^*_{j2} < Y^*_s < Y^*_{i1}$ for $i \neq j$. ■

Note that, since $Y^*_s < Y^*_{i1} = Y^*_i$, the optimal expected time of investment in this case is earlier than the optimal expected time of investment for each of the goods when they are independent.

Proposition 6 *Suppose $\eta > \gamma_A + \gamma_B$ and $\frac{I_i}{\gamma_i} < \frac{I_j}{\gamma_j}$ Given Y,*

1. *If $\frac{I_j}{(\eta - \gamma_i)} \leq \frac{I_i}{\gamma_i}$, then (P_s) is the optimal strategy and $Y^*_S \leq \min_i Y^*_i$*
2. *If $\frac{I_j}{(\eta - \gamma_i)} > \frac{I_i}{\gamma_i}$, then: (i) for $Y < Y^*_{j2}$,(P_{ij}) is the optimal strategy and $Y^*_{j2} < Y^*_j$; (ii) for $Y \geq Y^*_{j2}$, (P_s) is the optimal strategy*

Proof. See the Appendix ■

Note that, when the goods are complements, the case of $I_A/\gamma_A = I_B/\gamma_B$ corresponds to a high degree of complementarity, because (18) and (25) imply that $I_j/(\eta - \gamma_i) < I_j/\gamma_j = I_i/\gamma_i$ for $i \neq j$. Thus, the results in Propositions 5 and 6 are as our intuition would suggest, namely, invest simultaneously if there is a high degree of complementarity and invest in the appropriate sequential project otherwise. Furthermore, as our intuition would suggest, the expected time of simultaneous investment when there is a high degree of complementarity is no later than the expected time of investment for each of the goods when they are independent, and the expected time of stage two investment in the appropriate sequential project when there is a low degree of complementarity is earlier than in the case of independent goods.

6.2 Goods A and B are substitutes

Again, the results will depend on the degree of substitutability between the goods and the relationship between the investment costs and the cashflows. We have to consider the following possibilities: $\eta = \gamma_A = \gamma_B$; $\eta = \gamma_i > \gamma_j$ for $i \neq j$; $\eta > \gamma_i = \gamma_j$ for $i \neq j$; and $\eta > \gamma_i > \gamma_j$ for $i \neq j$.

The first two possibilities are extreme cases where the demand for one good completely wipes out the demand for the other good, so that, the instantaneous profit is maximized by selling only one good even when the monopolist has invested in both goods.

In what follows we state and provide the intuitions for the main results of the substitutes case. Given that the proofs are arithmetically involved, they are left for the appendix.

Proposition 7 *Suppose $\eta = \gamma_A = \gamma_B$. Then, given any Y:*

1. *If $I_A = I_B$, then the optimal strategy is to undertake either (P_{AB}) or (P_{BA}).*

2. *If $I_i < I_j$ for $i \neq j$, then (P_{ij}) is the optimal strategy.*

Proof. The first part follows from simple arithmetic. The second part follows from the following facts. First, $I_i < I_j$ implies $Y^*_{i1} < Y^*_{j1}$ and $PV_{ij}(Y) > PV_{ji}(Y)$ for all $Y \geq Y^*_{j1}$.Second, for any $Y < Y^*_{j1}$, since Y^*_{i1} is the critical point of investment in good i, the present discounted value of investing in good i at Y^*_{j1} is less than $PV_{ij}(Y)$. However, for any $Y < Y^*_{j1}$, the present discounted value of investing in good i at Y^*_{j1} is larger than

$PV_{ji}(Y)$ as $I_i < I_j$. So we also have $PV_{ij}(Y) > PV_{ji}(Y)$ for all $Y < Y_{j1}^*$. Hence, given any Y, the monopolist's optimal strategy is to undertake (P_{ij}). ■

The result is fairly intuitive. Given the high degree of substitutability between the goods, the simultaneous project (P_S) is suboptimal, and the sequential project (P_{ij}) entails investing only in good i at $\max\{Y, Y_{i1}^*\}$. Notice that in this case the cashflows of both projects are exactly the same. Therefore, the investment cost determines entirely which one to undertake for any Y. This should come as no surprise: without taking into account the initial investment, the monopolist, who only cares about maximizing profits, is indifferent between both projects. Therefore, she chooses the less expensive one, or flips a coin if the two cost the same.

Proposition 8 *Given any Y, suppose $\eta = \gamma_i > \gamma_j$ for $i \neq j$. If $Y_{i1}^* \leq Y_{j1}^*$, then it is optimal to undertake (P_{ij}). If $Y_{i1}^* > Y_{j1}^*$, then:*

1. *If $[\eta^{\beta_1} - (\eta - \gamma_j)^{\beta_1}] I_i^{1-\beta_1} > \gamma_j^{\beta_1} I_j^{1-\beta_1}$, then it is optimal to undertake (P_{ij});*
2. *If $[\eta^{\beta_1} - (\eta - \gamma_j)^{\beta_1}] I_i^{1-\beta_1} = \gamma_j^{\beta_1} I_j^{1-\beta_1}$, then: (i) it is optimal to undertake either (P_{ij}) or (P_{ji}) if $Y \leq Y_{j1}^*$; (ii) it is optimal to undertake (P_{ij}) if $Y > Y_{j1}^*$;*
3. *If $[\eta^{\beta_1} - (\eta - \gamma_j)^{\beta_1}] I_i^{1-\beta_1} < \gamma_j^{\beta_1} I_j^{1-\beta_1}$, then there exists $\hat{Y} \in (Y_{j1}^*, Y_{i2}^*)$ such that: (i) it is optimal to undertake (P_{ji}) if $Y < \hat{Y}$; (ii) it is optimal to undertake either (P_{ij}) or (P_{ji}) if $Y = \hat{Y}$; (iii) it is optimal to undertake (P_{ij}) if $Y > \hat{Y}$.*

Proof. See the appendix. ■

Again, the high degree of substitutability between the goods renders the simultaneous investment project suboptimal and the sequential project (P_{ij}) entails investing only in good i. As for the other sequential project, (P_{ji}), as aforementioned, it entails investing in good j, then exiting good j upon entry in good i.The conditions stated in the proposition determine which of the simultaneous projects will be actually undertaken.

Proposition 8 illustrates the cannibalism phenomena we discussed in the introduction. Without loss of generality, consider $i = A$ and $j = B$. If this is the case, Proposition 8 tells us that from the moment that the monopolist enters the production of good A, she should no longer produce good B. However, before the optimal time to enter the production of good A, there is demand for good B. If it is rentable to produce good B until it comes the time to invest in A, then (P_{ji}) will be the project to be carried out.

In what follows, we consider the less extreme cases of substitutability.

Proposition 9 *Suppose $\eta < \gamma_i + \gamma_j$ and $\eta > \gamma_i = \gamma_j$ for $i \neq j$. Then, given any Y:*

1. *If $I_i = I_j$, (i) both (P_{ij}) and (P_{ji}) are optimal if $Y < Y^*_{j2}$; (ii) (P_s) is optimal if $Y \geq Y^*_{j2}$.*
2. *If $I_i < I_j$, (i) (P_{ij}) is optimal if $Y < Y^*_{j2}$; (ii) (P_s) is optimal if $Y \geq Y^*_{j2}$.*

Proof. See the appendix. ■

Proposition 10 *Suppose $\eta < \gamma_A + \gamma_B$, $\eta > \gamma_i > \gamma_j$ and $Y^*_{i1} \leq Y^*_{j1}$ for $i \neq j$. Then, given any Y:*

1. *(P_{ij}) is optimal if $Y < Y^*_{j2}$;*
2. *(P_s) is optimal if $Y \geq Y^*_{j2}$.*

Proof. See the appendix. ■

Proposition 11 *Suppose $\eta < \gamma_A + \gamma_B$, $\eta > \gamma_i > \gamma_j$ and $Y^*_{i1} > Y^*_{j1}$ for $i \neq j$. If $Y^*_{i2} \geq Y^*_{j2}$, then:*

1. *(P_{ji}) is optimal for $Y < Y^*_{i2}$;*
2. *(P_s) is optimal for $Y \geq Y^*_{i2}$.*

Proof. See the Appendix ■

Proposition 12 *Suppose $\eta < \gamma_A + \gamma_B$ and $\eta > \gamma_i > \gamma_j$ for $i \neq j$. Also, suppose $Y^*_{i1} > Y^*_{j1}$ and $Y^*_{i2} < Y^*_{j2}$. Then:*

1. *If $[\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]I_i^{1-\beta_1} > [\gamma_j^{\beta_1} - (\eta - \gamma_i)^{\beta_1}]I_j^{1-\beta_1}$, then (P_{ij}) is optimal for $Y < Y^*_{j2}$ and (P_s) is optimal for $Y \geq Y^*_{j2}$;*
2. *If $[\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]I_i^{1-\beta_1} = [\gamma_j^{\beta_1} - (\eta - \gamma_i)^{\beta_1}]I_j^{1-\beta_1}$, then: (i) both (P_{ij}) and (P_{ji}) are optimal for $Y \leq Y^*_{j1}$, (ii) (P_{ij}) is optimal for $Y^*_{j1} < Y < Y^*_{j2}$, (iii) (P_s) is optimal for $Y \geq Y^*_{j2}$;*

3. *If* $[\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}] I_i^{1-\beta_1} < [\gamma_j^{\beta_1} - (\eta - \gamma_i)^{\beta_1}] I_j^{1-\beta_1}$*, then there exists* $\tilde{Y} \in (Y_{j1}^*, Y_{i2}^*)$ *such that: (i)* (P_{ji}) *is optimal for* $Y < \tilde{Y}$*, (ii) both* (P_{ij}) *and* (P_{ji}) *are optimal at* $Y = \tilde{Y}$*, (iii)* (P_{ij}) *is optimal for* $\tilde{Y} < Y < Y_{j2}^*$*, (iv)* (P_s) *is optimal for* $Y \geq Y_{j2}^*$.

Proof. See the appendix. ■

Note that, when the goods are substitutes, using (18) and (25), it can be easily verified that $Y_i^* < Y_{i2}^*$ for $i = A, B$. Therefore, whenever the goods are substitutes and it is optimal to undertake a sequential project, as our intuition would suggest, the optimal expected time of investment in stage one is the same as in the case of independent goods but the optimal expected time of investment in stage two is later than in the case of independent goods. Another fairly intuitive result is that, when the goods are substitutes, we observe that the simultaneous project (P_s) can only be optimal by default, i.e. only if $Y \geq \max\{Y_{A2}^*, Y_{B2}^*\}$.

7 Conclusion

In this paper, we proposed a framework for studying the optimal investment strategy of a monopolist who faces the problem of choosing to produce interrelated goods. We show that both the timing and the order (say invest in A and then in B) under which the monopolist will invest may completely differ from the textbook case for a given normalization. Under some circumstances, the ordering is inverted, in some cases the time at which the monopolist invest in A or the time in which invest in B changes. We found that the optimal strategy depends on the degree of substitutability between the goods and the initial investment costs.

In general, when the goods are substitutes, sequential investment is likely to be optimal. In this case, the timing of investment in good A is the same as the textbook timing when investment in goods A and B is independent. However, the entry threshold for good B is different from the textbook entry threshold.

When the goods are complements, the optimal investment strategy is likely to be simultaneous investment. Furthermore, the simultaneous timing of investment in both goods A and B differs from that when investment in each good is considered independently.

We hope that this paper helps building the bridge between the IO product segmentation literature and the real options one. We are conscious that there are many questions that are left unanswered, but they are out of the scope of this paper. For example, we have only considered the optimal entry problem of the monopolist. However, we could ask ourselves how the presence of other competing firms, considering entry, may affect the timing and the choice of which product to invest in. Additionally, we could consider if the existence of an opportunity to invest in a substitute market could serve the purpose of a commitment device in order to deter entry. These, and other questions, are left for further research.

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A Appendix

Proof of Proposition 4 Let $Y_{i1}^* < Y_{j2}^*$ for $i \neq j$. Using (18) and (25), we get $I_i/\gamma_i < I_j/(\eta - \gamma_i)$. Then it is easy to verify that $I_i/\gamma_i < (I_i + I_j)/\eta < I_j/(\eta - \gamma_i)$. So (18), (25) and (31) imply that $Y_{i1}^* < Y_s^* < Y_{j2}^*$.Let $Y < Y_{j2}^*$ and consider any y_i, y_j such that $Y \leq y_i \leq y_j < Y_{j2}^*$. Denote by $V(Y, y_i, y_j)$ the present discounted value of investing sequentially in good i at y_i and in good j at y_j. Since Y_{i1}^* and Y_{j2}^* are respectively the critical points of investment in goods i and j under (P_{ij}), it must be the case that $PV_{ij}(Y) > V(Y, y_i, y_j)$. It can also be checked that, if $y_i = y_j = \max\{Y, Y_s\}$, then $V(Y, y_i, y_j) = PV_s(Y)$. Thus, we have $PV_{ij}(Y) > PV_s(Y)$. •

Given any Y, in the remainder of this appendix let $\Delta(Y) = PV_{ij}(Y) - PV_{ji}(Y)$ for $i \neq j$.

Proof of Proposition 6 When $I_i/\gamma_i < I_j/\gamma_j$ for $i \neq j$, $\eta > \gamma_A + \gamma_B$ implies that $I_i/(\eta - \gamma_j) < I_i/\gamma_i < I_j/\gamma_j$. So (P_{ji}) is not an optimal strategy because $Y_{i2}^* < Y_{j1}^*$ according to (18), (25) and $I_i/(\eta - \gamma_j) < I_j/\gamma_j$. To determine the choice between the remaining two projects, we consider two possibilities depending on the degree of complementarity:

1. Suppose $I_j/(\eta - \gamma_i) \leq I_i/\gamma_i$, which corresponds to a high degree of complementarity. Then, as (18), (25) and (31) imply that $Y_{j2}^* \leq Y_s^* \leq Y_{i1}^*$, the optimal strategy is (P_s). In this case, since $Y_s^* \leq Y_{i1}^* = Y_i^* < Y_j^*$, the optimal expected time of investment is no later than the optimal expected time of investment for each of the goods when they are independent.
2. Now, suppose $I_j/(\eta - \gamma_i) > I_i/\gamma_i$, which corresponds to a lower degree of complementarity. Using (18), (25) and (31), it can be verified that $Y_{i1}^* < Y_s^* < Y_{j2}^*$. So, if $Y < Y_{j2}^*$, Proposition 4 implies that (P_{ij}) is the optimal strategy. In this case, since it can be checked that $Y_{j2}^* < Y_j^*$, the optimal expected time of investment in good i is the same as when the goods are independent but the optimal expected time of investment in good j is earlier than when the goods are independent. On the other hand, if $Y \geq Y_{j2}^*$, it is obvious that (P_s) is the optimal strategy by default.

Proof of Proposition 8 Let $\eta = \gamma_i > \gamma_j$ for $i \neq j$.

Suppose $Y_{i1}^* \leq Y_{j1}^* < Y_{i2}^*$[5]. We need to show that $\Delta(Y) > 0$ for every Y. Notice that $Y_{i1}^* \leq Y_{j1}^*$ implies that:

$$\left(\frac{I_i}{I_j}\right)^{\beta_1 - 1} \leq \left(\frac{\gamma_i}{\gamma_j}\right)^{\beta_1} - \left(\frac{\gamma_i - \gamma_j}{\gamma_j}\right)^{\beta_1} \tag{35}$$

Inequalities (37) and (38) below will follow from (35). In this case, we have that:

$$\Delta(Y) = \begin{cases} \left(\frac{Y(\beta_1-1)}{\beta_1(\rho-\mu)}\right)^{\beta_1} \frac{1}{\beta_1-1}\left[I_i^{1-\beta_1}\left(\gamma_i^{\beta_1} - (\gamma_i-\gamma_j)^{\beta_1}\right) - \gamma_j^{\beta_1} I_j^{1-\beta_1}\right] \text{ if } Y \leq Y_{i1}^* \\ \frac{\gamma_i Y}{\rho-\mu} - I_i - \left(\frac{Y(\beta_1-1)}{\beta_1(\rho-\mu)}\right)^{\beta_1} \frac{1}{\beta_1-1}\left[I_j^{1-\beta_1}\gamma_j^{\beta_1} + (\gamma_i-\gamma_j)^{\beta_1} I_i^{1-\beta_1}\right] \text{ if } Y_{i1}^* \leq Y \leq Y_{j1}^* \\ \left(\frac{(\gamma_i-\gamma_j)Y}{\rho-\mu} - I_i + I_j\right) - \left(\frac{Y}{Y_{i2}^*}\right)^{\beta_1}\left(\frac{(\gamma_i-\gamma_j)}{\rho-\mu}Y_{i2}^* - I_i\right) \text{ if } Y_{j1}^* \leq Y \leq Y_{i2}^* \\ I_j \text{ if } Y_{i2}^* \leq Y \end{cases} \tag{36}$$

For $Y \leq Y_{i1}^*$, (36) above implies that:

$$\Delta(Y) > 0 \tag{37}$$

where the last inequality follows from (35). When $Y_{i1}^* < Y_{j1}^*$, for $Y_{i1}^* \leq Y \leq Y_{j1}^*$, we get the second part of (36). Notice that:

$$\Delta(Y_{i1}^*) = \frac{I_i}{\beta_1 - 1}\left[\left(\gamma_i^{\beta_1} - (\gamma_i - \gamma_j)^{\beta_1}\right) - \left(\frac{I_i}{I_j}\right)^{\beta_1 - 1}\gamma_j^{\beta_1}\right] > 0 \tag{38}$$

where the inequality again follows from (35). What is more, it is straightforward to see that:

$$\frac{d}{dY}\Delta(Y) = \frac{1}{\rho - \mu}\left[\gamma_i - \left(\frac{Y}{Y_{j1}^*}\right)^{\beta_1 - 1}\gamma_j - (\gamma_i - \gamma_j)\left(\frac{Y}{Y_{i2}^*}\right)^{\beta_1 - 1}\right], Y \in \left(Y_{i1}^*, Y_{j1}^*\right) \tag{39}$$

[5] Notice that $Y_{i1}^* \leq Y_{j1}^*$ implies that:

$$\left(\frac{I_i}{I_j}\right)^{\beta_1 - 1} \leq \left(\frac{\gamma_i}{\gamma_j}\right)^{\beta_1} - \left(\frac{\gamma_i - \gamma_j}{\gamma_j}\right)^{\beta_1}$$

Inequalities (37) and (38) below follow from this.

Since $Y \leq Y_{j1}^* < Y_{i2}^*$, $\left(\frac{Y}{Y_{j1}^*}\right) \leq 1$ and $\left(\frac{Y}{Y_{i2}^*}\right) < 1$. Then, as $\beta_1 > 1$, it follows that:

$$\frac{d}{dY}\Delta\left(Y\right) > \frac{1}{\rho-\mu}\left[\gamma_i - \gamma_j - \left(\gamma_i - \gamma_j\right)\right] > 0 \tag{40}$$

Then, $\Delta\left(Y\right)$ is a strictly increasing function in $\left(Y_{i1}^*, Y_{j1}^*\right)$ and $\Delta\left(Y_{i1}^*\right) > 0$. Thus, $\Delta\left(Y\right) > 0$ for $Y \in \left[0, Y_{j1}^*\right]$ .For $Y_{j1}^* \leq Y < Y_{i2}^*$,we have that:

$$\frac{d\Delta(Y)}{dY} = \left(\frac{\gamma_i - \gamma_j}{\rho-\mu}\right)\left(1 - \left(\frac{Y}{Y_{i2}^*}\right)^{\beta_1 - 1}\right) > 0$$

where the last inequality follows from $Y < Y_{i2}^*$ and $\beta_1 > 1$. Hence, since we have already shown that $\Delta(Y_{j1}^*) > 0$, we have $\Delta(Y) > 0$ for $Y_{j1}^* \leq Y < Y_{i2}^*$. For $Y \geq Y_{i2}^*$, it is obvious from (36) that $\Delta(Y) = I_j > 0$.

Now, suppose $Y_{i1}^* > Y_{j1}^*$. Using (18), (25) and (34), it can be verified that:

$$\Delta(Y) = \begin{cases} (\beta_1 - 1)^{\beta_1 - 1}\left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1}\left[\frac{\eta^{\beta_1} - (\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1 - 1}} - \frac{\gamma_j^{\beta_1}}{I_j^{\beta_1 - 1}}\right] & \text{if } Y < Y_{j1}^* \\ (\beta_1 - 1)^{\beta_1 - 1}\left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1}\left[\frac{\eta^{\beta_1} - (\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1 - 1}}\right] - \frac{\gamma_j Y}{\rho-\mu} + I_j & \text{if } Y_{j1}^* \leq Y < Y_{i1}^* \\ \frac{(\eta-\gamma_j)Y}{\rho-\mu} - I_i + I_j - \left[\frac{(\eta-\gamma_j)Y_{i2}^*}{\beta_1(\rho-\mu)}\right]\left(\frac{Y}{Y_{i2}^*}\right)^{\beta_1} & \text{if } Y_{i1}^* \leq Y < Y_{i2}^* \\ I_j & \text{if } Y \geq Y_{i2}^* \end{cases}$$

Then we consider each of the three possibilities stated in the proposition in turn:

1. Suppose $[\eta^{\beta_1} - (\eta-\gamma_j)^{\beta_1}]I_i^{1-\beta_1} > \gamma_j^{\beta_1} I_j^{1-\beta_1}$. It can be checked that $d\Delta(Y)/dY > 0$ and $d^2\Delta(Y)/dY^2 > 0$ for $Y < Y_{j1}^*$, and $d^2\Delta(Y)/dY^2 > 0$ for $Y_{j1}^* < Y < Y_{i1}^*$. So it must be the case that $d\Delta(Y)/dY > 0$ for $Y < Y_{i1}^*$ as $\Delta(Y)$ is differentiable. Hence, $\Delta(Y) > 0$ for $0 < Y \leq Y_{i1}^*$. Also, it is easy to check that $d\Delta(Y)/dY > 0$ for $Y_{i1}^* < Y < Y_{i2}^*$, which implies that $\Delta(Y) > 0$ for $Y_{i1}^* < Y < Y_{i2}^*$. For $Y \geq Y_{i2}^*$, we have $\Delta(Y) = I_j > 0$.
2. Suppose $[\eta^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]I_i^{1-\beta_1} = \gamma_j^{\beta_1} I_j^{1-\beta_1}$. Then it can be verified that $d\Delta(Y)/dY = 0$ for $Y < Y_{j1}^*$ and $d\Delta(Y)/dY > 0$ for $Y_{j1}^* < Y < Y_{i2}^*$. Hence, it must be the case that $\Delta(Y) = 0$ for $Y \leq Y_{j1}^*$ and $\Delta(Y) > 0$ for $Y_{j1}^* < Y < Y_{i2}^*$. For $Y \geq Y_{i2}^*$, we have $\Delta(Y) = I_j > 0$.

3. Suppose $[\eta^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]I_i^{1-\beta_1} < \gamma_j^{\beta_1} I_j^{1-\beta_1}$. Then it can be checked that $d\Delta(Y)/dY < 0$ and $d^2\Delta(Y)/dY^2 < 0$ for $Y < Y_{j1}^*$, $d^2\Delta(Y)/dY^2 > 0$ for $Y_{j1}^* < Y < Y_{i1}^*$ and $d\Delta(Y)/dY > 0$ for $Y_{i1}^* < Y < Y_{i2}^*$. We also know that $\Delta(Y)$ is differentiable and $\Delta(Y) = I_j > 0$ for $Y \geq Y_{i2}^*$. Thus, it must be the case that there exists a $\hat{Y} \in (Y_{j1}^*, Y_{i2}^*)$ such that $\Delta(Y) < 0$ for $0 < Y < \hat{Y}$, $\Delta(\hat{Y}) = 0$, and $\Delta(Y) > 0$ for $Y > \hat{Y}$.

Proof of Proposition 9: Let $\eta < \gamma_i + \gamma_j$ and $\eta > \gamma_i = \gamma_j$ for $i \neq j$. Suppose $I_i = I_j$. Using (18) and (25), it is straightforward to verify that $Y_{i1}^* = Y_{j1}^* < Y_{i2}^* = Y_{j2}^*$. So Proposition 4 implies that the simultaneous project (P_s) is suboptimal for any $Y < Y_{i2}^*$. Also, (34) implies that $PV_{ij}(Y) = PV_{ji}(Y)$ for $Y < Y_{j2}^*$. Thus, for $Y < Y_{j2}^*$, both (P_{ij}) and (P_{ji}) are optimal projects. On the otherhand, for $Y \geq Y_{j2}^*$, (P_s) is optimal by default.

Now, suppose $I_i < I_j$. Using (18) and (25), it can be checked that $Y_{i1}^* < Y_{i2}^* < Y_{j2}^*$. So Proposition 4 implies that the sequential project (P_{ij}) dominates the simultaneous project (P_s) for $Y < Y_{j2}^*$. Since it is meaningless to consider the project (P_{ji}) if $Y_{j1}^* \geq Y_{i2}^*$, we will assume that $Y_{j1}^* < Y_{i2}^*$. We show below that $PV_{ij}(Y) > PV_{ji}(Y)$ for $Y < Y_{j2}^*$. Also, as observed in the main body, the simultaneous project (P_s) is optimal by default if $Y \geq Y_{j2}^*$. Thus, (P_{ij}) is optimal if $Y < Y_{j2}^*$ and (P_s) is optimal if $Y \geq Y_{j2}^*$.

Now, to complete the proof, we only need to show that $\Delta(Y) > 0$ for $Y < Y_{j2}^*$ if $I_i < I_j$ and $Y_{j1}^* < Y_{i2}^*$. So suppose $I_i < I_j$ and $Y_{j1}^* < Y_{i2}^*$. Then it can be easily verified that $Y_{i1}^* < Y_{j1}^* < Y_{i2}^* < Y_{j2}^*$. Since we already know that $PV_{ji}(Y) = PV_s(Y)$ for $Y \geq Y_{i2}^*$, Proposition 4 implies that $\Delta(Y) > 0$ for $Y \geq Y_{i2}^*$. Consider any $Y < Y_{i2}^*$ and let $\bar{V}$ be the present discounted value of investing sequentially in good i at $\max\{Y, Y_{j1}^*\}$ and in good j at Y_{i2}^*, i.e.

$$\bar{V} = \begin{cases} \left(\frac{Y}{Y_{j1}^*}\right)^{\beta_1} \left[\frac{\gamma_i Y_{j1}^*}{\rho-\mu} - I_i\right] + \left(\frac{Y}{Y_{i2}^*}\right)^{\beta_1} \left[\frac{(\eta-\gamma_i)Y_{i2}^*}{\rho-\mu} - I_j\right] & \text{if } Y < Y_{j1}^* \\ \frac{\gamma_i Y}{\rho-\mu} - I_i + \left(\frac{Y}{Y_{i2}^*}\right)^{\beta_1} \left[\frac{(\eta-\gamma_i)Y_{i2}^*}{\rho-\mu} - I_j\right] & \text{if } Y_{j1}^* \leq Y < Y_{i2}^* \end{cases}$$

Since Y_{i1}^* and Y_{j2}^* are the critical points of (P_{ij}), it is obvious that $PV_{ij}(Y) > \bar{V}$, and hence, $\Delta(Y) > \bar{V} - PV_{ji}(Y)$. Using (34), $\gamma_i = \gamma_j$ and $I_i < I_j$, it can also be verified that $\bar{V} - PV_{ji}(Y) > 0$. Thus, we have $\Delta(Y) > 0$.

Proof of Proposition 10 Let $\eta < \gamma_A + \gamma_B$, $\eta > \gamma_i > \gamma_j$ and $Y_{i1}^* \leq Y_{j1}^*$ for $i \neq j$. Then $(\gamma_i + \gamma_j - \eta)(\gamma_i - \gamma_j) > 0$, which implies that $(\eta - \gamma_j)/(\eta - \gamma_i) > \gamma_i/\gamma_j$. Also, since (25) and $Y_{i1}^* \leq Y_{j1}^*$ imply that $I_i/I_j \leq \gamma_i/\gamma_j$, we get $(\eta - \gamma_j)/(\eta - \gamma_i) > I_i/I_j$. It then follows from (18) that $Y_{i2}^* < Y_{j2}^*$. By (18), (25) and $\eta < \gamma_i + \gamma_j$, we also have $Y_{i1}^* < Y_{i2}^*$, and hence, $Y_{i1}^* < Y_{i2}^* < Y_{j2}^*$. To complete the proof, we need to show that $\Delta(Y) > 0$ for $Y < Y_{j2}^*$ if $Y_{i1}^* \leq Y_{j1}^* < Y_{i2}^* < Y_{j2}^*$. Suppose $Y_{i1}^* \leq Y_{j1}^* < Y_{i2}^* < Y_{j2}^*$. Since we already know that $PV_{ji}(Y) = PV_s(Y)$ for $Y \geq Y_{i2}^*$, Proposition 4 implies that $\Delta(Y) > 0$ for $Y_{i2}^* \leq Y < Y_{j2}^*$. Now, consider any $Y \leq Y_{i2}^*$. Using (18), (25) and (34), it can be verified that

$$\Delta(Y) = \begin{cases} (\beta_1 - 1)^{\beta_1 - 1} \left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1} \left[\frac{\gamma_i^{\beta_1} - (\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} - \frac{\gamma_j^{\beta_1} - (\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y \leq Y_{i1}^* \\ \frac{\gamma_i Y}{\rho-\mu} - I_i - (\beta_1 - 1)^{\beta_1 - 1} \left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1} \left[\frac{(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} + \frac{\gamma_j^{\beta_1} - (\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y_{i1}^* < Y \leq Y_{j1}^* \\ \frac{(\gamma_i-\gamma_j)Y}{\rho-\mu} - I_i + I_j + (\beta_1 - 1)^{\beta_1 - 1} \left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1} \left[\frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}} - \frac{(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}}\right] & \text{if } Y_{j1}^* < Y \leq Y_{i2}^* \end{cases}$$

Then we have

$$\frac{d\Delta(Y)}{dY} = \left[\frac{(\beta_1 - 1)Y}{\beta_1}\right]^{\beta_1 - 1} \left(\frac{1}{\rho - \mu}\right)^{\beta_1} \left[\frac{\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} - \frac{\gamma_j^{\beta_1} - (\eta - \gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] \quad \text{for } Y \leq Y_{i1}^*$$

$$\frac{d^2\Delta(Y)}{dY^2} = \begin{cases} \left[\frac{Y^{\beta_1-2}}{\beta_1^{\beta_1-1}}\right] \left(\frac{\beta_1-1}{\rho-\mu}\right)^{\beta_1} \left[\frac{\gamma_i^{\beta_1} - (\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} - \frac{\gamma_j^{\beta_1} - (\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y < Y_{i1}^* \\ -\left[\frac{Y^{\beta_1-2}}{\beta_1^{\beta_1-1}}\right] \left(\frac{\beta_1-1}{\rho-\mu}\right)^{\beta_1} \left[\frac{(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} + \frac{\gamma_j^{\beta_1} - (\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y_{i1}^* < Y < Y_{j1}^* \\ \left[\frac{Y^{\beta_1-2}}{\beta_1^{\beta_1-1}}\right] \left(\frac{\beta_1-1}{\rho-\mu}\right)^{\beta_1} \left[\frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}} - \frac{(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}}\right] & \text{if } Y_{j1}^* < Y < Y_{i2}^* \end{cases}$$

Since $\gamma_i > \gamma_j$ and $(\eta - \gamma_j)/(\eta - \gamma_i) > I_i/I_j$, it can be verified that $d^2\Delta(Y)/dY^2 < 0$ for $Y_{j1}^* < Y < Y_{i2}^*$. Also, if $Y_{i1}^* < Y_{j1}^*$, then $\gamma_i + \gamma_j > \eta$ implies that $d^2\Delta(Y)/dY^2 < 0$ for $Y_{i1}^* < Y < Y_{j1}^*$. Now, suppose $[\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]/I_i^{\beta_1-1} \leq \gamma_j^{\beta_1} - (\eta - \gamma_i)^{\beta_1}/I_j^{\beta_1-1}$. Then it is easy to check that $\Delta(Y) \leq 0$, $d\Delta(Y)/dY \leq 0$ and $d^2\Delta(Y)/dY^2 \leq 0$ for $Y < Y_{i1}^*$. Hence, because $\Delta(Y)$ is differentiable, we get $\Delta(Y_{i2}^*) < 0$, which we know is false. So it must be the case that $[\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]/I_i^{\beta_1-1} > \gamma_j^{\beta_1} - (\eta - \gamma_i)^{\beta_1}/I_j^{\beta_1-1}$. Then we have $\Delta(Y) > 0$, $d\Delta(Y)/dY > 0$ and $d^2\Delta(Y)/dY^2 > 0$ for $0 < Y < Y_{i1}^*$.

Thus, the differentiability of $\Delta(Y)$ and $\Delta(Y_{i2}^*) > 0$ also imply that $\Delta(Y) > 0$ for $Y_{i1}^* \leq Y \leq Y_{i2}^*$.

Proof of Proposition 11: Let $\eta < \gamma_i + \gamma_j$, $\eta > \gamma_i > \gamma_j$, $Y_{i1}^* > Y_{j1}^*$ and $Y_{i2}^* \geq Y_{j2}^*$ for $i \neq j$. We need to show that $\Delta(Y) < 0$ for $Y < Y_{i2}^*$ if $Y_{i1}^* < Y_{j2}^*$. Since we already know that $PV_{ij}(Y) = PV_s(Y)$ for $Y \geq Y_{j2}^*$, Proposition 4 implies that $\Delta(Y) < 0$ for $Y_{j2}^* \leq Y < Y_{i2}^*$ if $Y_{j2}^* < Y_{i2}^*$.

Now, consider any $Y \leq Y_{j2}^*$. Using (18), (25) and (34), it can be verified that:

$$\Delta(Y) = \begin{cases} (\beta_1 - 1)^{\beta_1 - 1} \left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1} \left[\frac{\gamma_i^{\beta_1} - (\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} - \frac{\gamma_j^{\beta_1} - (\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y \leq Y_{j1}^* \\ -\frac{\gamma_j Y}{\rho-\mu} + I_j + (\beta_1 - 1)^{\beta_1 - 1} \left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1} \left[\frac{\gamma_i^{\beta_1} - (\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} + \frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y_{j1}^* < Y \leq Y_{i1}^* \\ \frac{(\gamma_i-\gamma_j)Y}{\rho-\mu} - I_i + I_j + (\beta_1 - 1)^{\beta_1 - 1} \left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1} \left[\frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}} - \frac{(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}}\right] & \text{if } Y_{i1}^* < Y \leq Y_{j2}^* \end{cases}$$

Then we have:

$$\frac{d\Delta(Y)}{dY} = \begin{cases} \left[\frac{(\beta_1-1)Y}{\beta_1}\right]^{\beta_1-1} \left(\frac{1}{\rho-\mu}\right)^{\beta_1} \left[\frac{\gamma_i^{\beta_1} - (\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} - \frac{\gamma_j^{\beta_1} - (\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y \leq Y_{j1}^* \\ \left(\frac{\gamma_i-\gamma_j}{\rho-\mu}\right) + \left[\frac{(\beta_1-1)Y}{\beta_1}\right]^{\beta_1-1} \left(\frac{1}{\rho-\mu}\right)^{\beta_1} \left[\frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}} - \frac{(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}}\right] & \text{if } Y_{i1}^* < Y \leq Y_{j2}^* \end{cases}$$

$$\frac{d^2\Delta(Y)}{dY^2} = \begin{cases} \left[\frac{Y^{\beta_1-2}}{\beta_1^{\beta_1-1}}\right] \left(\frac{\beta_1-1}{\rho-\mu}\right)^{\beta_1} \left[\frac{\gamma_i^{\beta_1} - (\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} - \frac{\gamma_j^{\beta_1} - (\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y < Y_{j1}^* \\ \left[\frac{Y^{\beta_1-2}}{\beta_1^{\beta_1-1}}\right] \left(\frac{\beta_1-1}{\rho-\mu}\right)^{\beta_1} \left[\frac{\gamma_i^{\beta_1} - (\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} + \frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y_{j1}^* < Y < Y_{i1}^* \\ \left[\frac{Y^{\beta_1-2}}{\beta_1^{\beta_1-1}}\right] \left(\frac{\beta_1-1}{\rho-\mu}\right)^{\beta_1} \left[\frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}} - \frac{(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}}\right] & \text{if } Y_{i1}^* < Y < Y_{j2}^* \end{cases}$$

Using (18), it can be verified that, at $Y = Y_{j2}^*$, $\Delta(Y) = d\Delta(Y)/dY = 0$ if $Y_{j2}^* = Y_{i2}^*$ and $d\Delta(Y)/dY > 0$ if $Y_{j2}^* < Y_{i2}^*$. Also, $\gamma_i + \gamma_j > \eta$ implies that $d^2\Delta(Y)/dY^2 > 0$ for $Y_{j1}^* < Y < Y_{i1}^*$. Furthermore, it is obvious that $d^2\Delta(Y)/dY^2$ is always positive or always negative on the interval (Y_{i1}^*, Y_{j2}^*). Now, suppose $[\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]/I_i^{\beta_1-1} \geq \gamma_j^{\beta_1} - (\eta - \gamma_i)^{\beta_1}/I_j^{\beta_1-1}$. Then it is easy to check that $\Delta(Y) \geq 0$, $d\Delta(Y)/dY \geq 0$ and $d^2\Delta(Y)/dY^2 \geq 0$ for $Y < Y_{j1}^*$. Hence, because $\Delta(Y)$ is differentiable, we get $\Delta(Y_{i1}^*) > 0$ and $d\Delta(Y)/dY > 0$ at $Y = Y_{i1}^*$. Thus, $d^2\Delta(Y)/dY^2$ must assume both positive and negative values on the interval (Y_{i1}^*, Y_{j2}^*), which is a contradiction. So it must be the case that $[\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]/I_i^{\beta_1-1} < \gamma_j^{\beta_1} - (\eta - \gamma_i)^{\beta_1}/I_j^{\beta_1-1}$, which implies that $\Delta(Y) < 0$ for $0 < Y \leq Y^*_{j1}$. Suppose $\Delta(Y^*_{i1}) \geq 0$. Then $\Delta(Y^*_{j1}) < 0$ and $d^2\Delta(Y)/dY^2 > 0$ for $Y^*_{j1} < Y < Y^*_{i1}$ imply that $d\Delta(Y)/dY > 0$ at $Y = Y^*_{i1}$. So it follows from $d^2\Delta(Y)/dY^2$ being always positive or always negative on the interval (Y^*_{i1}, Y^*_{j2}) and $\Delta(Y^*_{j2}) \leq 0$ that $d\Delta(Y)/dY < 0$ at $Y = Y^*_{j2}$, which is a contradiction. Hence, it must be the case that $\Delta(Y^*_{i1}) < 0$, which together with $\Delta(Y^*_{j1}) < 0$ and $d^2\Delta(Y)/dY^2 > 0$ for $Y^*_{j1} < Y < Y^*_{i1}$ imply that $\Delta(Y) < 0$ for $Y^*_{j1} < Y \leq Y^*_{i1}$. Finally, since $d^2\Delta(Y)/dY^2$ is always positive or always negative on the interval (Y^*_{i1}, Y^*_{j2}), $\Delta(Y^*_{j2}) \leq 0$, $\Delta(Y^*_{i1}) < 0$ and $d\Delta(Y)/dY \geq 0$ at $Y = Y^*_{j2}$ imply that $\Delta(Y) < 0$ for $Y^*_{i1} < Y < Y^*_{j2}$.

Proof of Proposition 12 Let $\eta < \gamma_i + \gamma_j$, $\eta > \gamma_i > \gamma_j$, $Y^*_{i1} > Y^*_{j1}$ and $Y^*_{i2} < Y^*_{j2}$ for $i \neq j$. So we have $Y^*_{j1} < Y^*_{i1} < Y^*_{i2} < Y^*_{j2}$. For $Y^*_{i2} \leq Y < Y^*_{j2}$, since $PV_{ji}(Y) = PV_s(Y)$, Proposition 4 implies that $\Delta(Y) > 0$. Using (18), (25) and (34), it can be verified that, for any $Y \leq Y^*_{i2}$,

$$\Delta(Y) = \begin{cases} (\beta_1-1)^{\beta_1-1}\left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1}\left[\frac{\gamma_i^{\beta_1}-(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} - \frac{\gamma_j^{\beta_1}-(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y \leq Y^*_{j1} \\ -\frac{\gamma_j Y}{\rho-\mu} + I_j + (\beta_1-1)^{\beta_1-1}\left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1}\left[\frac{\gamma_i^{\beta_1}-(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} + \frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y^*_{j1} < Y \leq Y^*_{i1} \\ \frac{(\gamma_i-\gamma_j)Y}{\rho-\mu} - I_i + I_j + (\beta_1-1)^{\beta_1-1}\left[\frac{Y}{\beta_1(\rho-\mu)}\right]^{\beta_1}\left[\frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}} - \frac{(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}}\right] & \text{if } Y^*_{i1} < Y \leq Y^*_{i2} \end{cases}$$

Then we have:

$$\frac{d\Delta(Y)}{dY} = \left[\frac{(\beta_1-1)Y}{\beta_1}\right]^{\beta_1-1}\left(\frac{1}{\rho-\mu}\right)^{\beta_1}\left[\frac{\gamma_i^{\beta_1}-(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} - \frac{\gamma_j^{\beta_1}-(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] \quad \text{for } Y \leq Y^*_{j1}$$

$$\frac{d^2\Delta(Y)}{dY^2} = \begin{cases} \left[\frac{Y^{\beta_1-2}}{\beta_1^{\beta_1-1}}\right]\left(\frac{\beta_1-1}{\rho-\mu}\right)^{\beta_1}\left[\frac{\gamma_i^{\beta_1}-(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} - \frac{\gamma_j^{\beta_1}-(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y < Y^*_{j1} \\ \left[\frac{Y^{\beta_1-2}}{\beta_1^{\beta_1-1}}\right]\left(\frac{\beta_1-1}{\rho-\mu}\right)^{\beta_1}\left[\frac{\gamma_i^{\beta_1}-(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}} + \frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}}\right] & \text{if } Y^*_{j1} < Y < Y^*_{i1} \\ \left[\frac{Y^{\beta_1-2}}{\beta_1^{\beta_1-1}}\right]\left(\frac{\beta_1-1}{\rho-\mu}\right)^{\beta_1}\left[\frac{(\eta-\gamma_i)^{\beta_1}}{I_j^{\beta_1-1}} - \frac{(\eta-\gamma_j)^{\beta_1}}{I_i^{\beta_1-1}}\right] & \text{if } Y^*_{i1} < Y < Y^*_{i2} \end{cases}$$

It can be verified that $\eta < \gamma_i + \gamma_j$ and $Y^*_{j1} < Y^*_{i1} < Y^*_{i2} < Y^*_{j2}$ imply $d^2\Delta(Y)/dY^2 > 0$ for $Y^*_{j1} < Y < Y^*_{i1}$ and $d^2\Delta(Y)/dY^2 < 0$ for $Y^*_{i1} < Y < Y^*_{i2}$. Then we consider each of the three possibilities stated in the proposition in turn.

1. Suppose $[\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]I_i^{1-\beta_1} > \gamma_j^{\beta_1 - (\eta-\gamma_i)^{\beta_1}} I_j^{1-\beta_1}$. In this case we need to show that $\Delta(Y) > 0$ for $0 < Y < Y_{i2}^*$. It can be checked that $d\Delta(Y)/dY > 0$ and $d^2\Delta(Y)/dY^2 > 0$ for $Y < Y_{j1}^*$. Then $d^2\Delta(Y)/dY^2 > 0$ for $Y_{j1}^* < Y < Y_{i1}^*$ implies that $\Delta(Y) > 0$ for $0 < Y \leq Y_{i1}^*$. Furthermore, $\Delta(Y_{i1}^*) > 0$, $\Delta(Y_{i2}^*) > 0$ and $d^2\Delta(Y)/dY^2 < 0$ for $Y_{i1}^* < Y < Y_{i2}^*$ imply that $\Delta(Y) > 0$ for $Y_{i1}^* < Y < Y_{i2}^*$.
2. Suppose $[\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]I_i^{1-\beta_1} = \gamma_j^{\beta_1 - (\eta-\gamma_i)^{\beta_1}} I_j^{1-\beta_1}$. Then we have $\Delta(Y) = d\Delta(Y)/dY = 0$ for $Y \leq Y_{j1}^*$. So we need to show that $\Delta(Y) > 0$ for $Y_{j1}^* < Y < Y_{i2}^*$. It can be checked that $\Delta(Y) = d\Delta(Y)/dY = 0$ for $Y \leq Y_{j1}^*$ and $d^2\Delta(Y)/dY^2 > 0$ for $Y_{j1}^* < Y < Y_{i1}^*$ imply that $\Delta(Y) > 0$ for $Y_{j1}^* < Y \leq Y_{i1}^*$. Also, $\Delta(Y_{i1}^*) > 0$, $\Delta(Y_{i2}^*) > 0$ and $d^2\Delta(Y)/dY^2 < 0$ for $Y_{i1}^* < Y < Y_{i2}^*$ imply that $\Delta(Y) > 0$ for $Y_{i1}^* < Y < Y_{i2}^*$.
3. Suppose $[\gamma_i^{\beta_1} - (\eta - \gamma_j)^{\beta_1}]I_i^{1-\beta_1} < \gamma_j^{\beta_1 - (\eta-\gamma_i)^{\beta_1}} I_j^{1-\beta_1}$. Then we have $\Delta(Y) < 0$ for $0 < Y \leq Y_{j1}^*$. Now, using $\Delta(Y_{j1}^*) < 0$, $\Delta(Y_{i2}^*) > 0$, $d^2\Delta(Y)/dY^2 > 0$ for $Y_{j1}^* < Y < Y_{i1}^*$, $d^2\Delta(Y)/dY^2 < 0$ for $Y_{i1}^* < Y < Y_{i2}^*$ and the differentiability of $\Delta(Y)$, it can be verified that there exists $\tilde{Y} \in (Y_{j1}^*, Y_{i2}^*)$ such that $\Delta(Y) < 0$ for $Y_{j1}^* < Y < \tilde{Y}$, $\Delta(\tilde{Y}) = 0$ and $\Delta(Y) > 0$ for $\tilde{Y} < Y < Y_{i2}^*$.