Extensive-Form Games and Strategic Complementarities

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Abstract

I prove the subgame-perfect equivalent of the basic result for Nash equilibria in normal-form games of strategic complements: the set of subgame-perfect equilibria is a non-empty, complete lattice—in particular, subgame-perfect Nash equilibria exist. For this purpose I introduce a device that allows the study of the set of subgame-perfect equilibria as the set of fixed points of a correspondence. My results are limited because extensive-form games of strategic complementarities turn out—surprisingly—to be a very restrictive class of games.

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1 Introduction

In this paper I define extensive-form games of strategic complementarities, and prove the subgame-perfect equivalent of the basic result for Nash equilibria in normal-form games of strategic complements: the set of subgame-perfect Nash equilibria (SPNE) is a non-empty, complete lattice. This has strong implications; not only does it give a general existence proof, it also allows the use of comparative statics techniques. While this seems to be a promising result, I also show that, in extensive-form games, the assumption of strategic complementarities is—surprisingly—very restrictive.

Equilibria are usually analyzed by means of fixed-point methods. This has not been the case for SPNE. A methodological contribution of this paper is the introduction of a device, the "extended best-response correspondence." with the property that the set of SPNE of a game coincides with the set of fixed points of the extended best-response correspondence. The model of extensive-form games that I use allows time to be continuous, so the extended best-response correspondence can also be used to analyze SPNE of continuous-time games.

SPNE exist in finite games. Harris, Reny, and Robson (1995) present an example of a game without an SPNE that is a two-stage game with a finite number of players, and where only one player has an infinite strategy space. Hence, existence of SPNE is not guaranteed after a minimal departure from finite games. Proofs of existence of SPNE in non-finite games are provided by Harris, Reny, and Robson (1995) for games of almost-perfect information where a public randomization device is present, by Harris (1985a), Harris (1985b) and Hellwig and Leininger (1987) for games of perfect information; and by Fudenberg and Levine (1983) for classes of games with strong "continuity at infinity" properties.

In this paper I show that the existence of SPNE follows from strategic complementar-

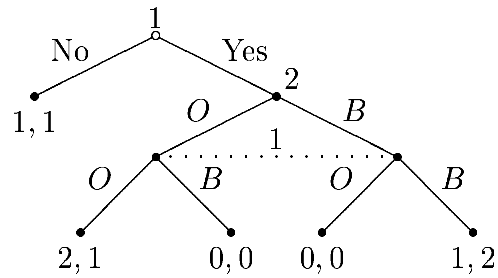

Figure 1: Optional Battle of the Sexes

ities; concretely, that the set of SPNE of a game whose normal-form is a game of strategic complementarities, is a non-empty, complete lattice. My results apply to continuous-time games and to games of imperfect information that are not necessarily games of almost-perfect information (I am not aware of other existence results for continuous-time games). The important problem with my results is that the assumption of complementarities in extensive-form games is very strong.

Games of strategic complementarities were first studied by Topkis (1979), and introduced into economics by Vives (1990). There are many examples of economic models that are games of strategic complements (see Milgrom and Roberts (1990), Topkis (1998), and Vives (1999)). By now it is fair to say that complementarities in normal-form games is a very useful and common structure. Here I show that, while still very useful, complementarities in dynamic contexts are rare.

To illustrate the problem, consider the game in Figure 1. This is "Optional Battle of the Sexes." Here, player One chooses first to say Yes or No. If One says No then payoffs are 1 each. If One says Yes then they play a Battle of the Sexes game: they simultaneously choose an element of $\{O, B\}$. If the choice is (O, O) then player One gets 2 while Two gets 1. If they choose (O, B) or (B, O) then both get a payoff of 0. If the outcome is (B, B) then payoffs are $(1, 2)$

It is easy to see that Battle of the Sexes (BoS, the simultaneous-move game that follows after One chooses Yes) is a game of strategic complementarities. Player One's best response to Two playing B is to play B and One's response to Two playing O is to play O. So, a change by Two from B to O makes One change in the same direction. This is also true for player Two: a change by One from B to O makes Two change in the same direction. Imposing an order on the players' strategies, we can say that O is "larger" than B. Then the best response of each player is increasing in the other player's choice of strategy, this is the crucial property of a game of strategic complementarities (indeed it is easily seen that BoS satisfies the definition of a game of strategic complementarities in e.g. Milgrom and Roberts (1990)).

Now, consider the extensive-form game Optional BoS and let us impose an order on the set $\{\text{No}, \text{Yes}\}$. Let the action "Yes" at One's initial decision node be larger than "No". Then the strategy No-O (say No at the initial node and plan to play O in Battle of the Sexes) is smaller than Yes-O and No-B is smaller than Yes-B. But, when One is playing No-B it is optimal for Two to play O, while if we increase One's strategy to Yes-B then it is uniquely optimal for Two to play B. This implies that Two's strategy is not increasing in One's strategy choice. We could try to fix this by saying that B is larger than O, but then the problem would arise when One increases the strategy from No-O to Yes-O.

It turns out that it is possible to make Optional BoS a game of strategic complementarities. The solution is to say that the action Yes is smaller than No. This shows that extensive-form games of strategic complementarities are not trivial. [1] But unfortunately the simple solution in Optional BoS is not feasible in general. I shall show how a complication of Optional BoS yields a game that cannot be transformed into a game of strategic complementarities. I shall argue also that most dynamic games of economic interest cannot be transformed into games of strategic complementarities.

[1] It is not true that they must be dominance-solvable, as might be suggested by the discussion above.

The situation contrasts with the study of Markov-Perfect equilibria in stochastic games with complementarities (Curtat (1996), Amir (1989) and Amir (1996)). Markov strategies limit dynamic strategic interactions, and thus allows complementarities to have leverage, but when more general strategies are allowed, this breaks down. I shall illustrate the situation with examples in Section 4.

Section 2 presents definitions and notation. Section 3 introduces the extended best-response correspondence and the main results of the paper. Section 4 show how complementarities are a restrictive assumption by discussing some examples.

2 Generalized Extensive-Form Games

2.1 Basic Definitions and Notation

A detailed discussion of the concepts defined in this subsection is in Topkis (1998). A set X with a transitive, reflexive, antisymmetric binary relation $\preceq$ is a ***lattice*** if whenever $x, y \in X$, both $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$ exist in X. It is ***complete*** if, for every nonempty subset A of X, $\inf A, \sup A$ exist in X. For two subsets A, B of X, say that A is smaller than B in the ***strong set order*** if $a \in A$, $b \in B$ implies $a \wedge b \in A, a \vee b \in B$. The ***order-interval topology*** on a lattice is obtained by taking the closed intervals $[x, y] = \{z \in X : x \preceq z \preceq y\}$ as a sub-basis for the closed sets. All lattices in the paper will be endowed with the order-interval topology. All products of partially ordered sets are endowed with the product order. All products of topological spaces are endowed with the product topology.

If X is a lattice, a function $f : X \to \mathbf{R}$ is ***quasisupermodular*** if for any $x, y \in X$, $f(x) \geq f(x \wedge y)$ implies $f(x \vee y) \geq f(y)$ and $f(x) > f(x \wedge y)$ implies $f(x \vee y) > f(y)$. [2] Let T be a partially ordered set. A function $f : X \times T \to \mathbf{R}$ satisfies the ***single-***

[2] Quasisupermodularity is an ordinal notion of complementarities, it was introduced by Milgrom and Shannon (1994).

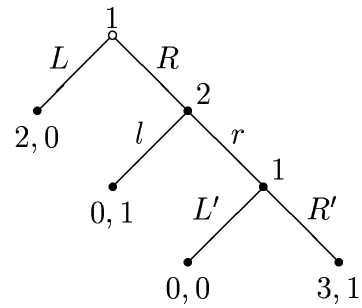

Figure 2: An extensive-form game of strategic complementarities

crossing condition in (x, t) if whenever $x \prec x'$ and $t \prec t'$, $f(x,t) \leq f(x',t)$ implies that $f(x,t') \leq f(x',t')$ and $f(x,t) < f(x',t)$ implies that $f(x,t') < f(x',t')$. The restriction of a function $f : X \to Y$ to a subset $X' \subseteq X$ is denoted $f|_{X'}$.

2.2 Definition of Generalized Extensive-Form Games

I present a definition of extensive-form games that has information sets, as opposed to decision nodes, as primitives. It is really only a slight variation on the usual rules for drawing game trees, but it results in a considerably more general framework because it allows time to be continuous and does not impose perfect recall or partitioned information structures. [3] I hope that the benefits of having results that apply to continuous-time games are important enough to balance the cost of a slightly unfamiliar framework. Besides its generality, this model of extensive-form games is more parsimonious than the usual one, therefore the proofs are easier and sharper than they would be otherwise.

I shall use the simple game in Figure 2 to illustrate the concepts as they are introduced. The game can be described as follows. First, player One selects an element in $\{L, R\}$. If she selects L then the game "ends" and the payoffs are 2 to player One and 0 to player Two. If she selects R then Two gets to choose between l and r. If he chooses l then she gets 1 while One gets 0. If he chooses r then player One gets to choose an element in

[3] See Fudenberg and Tirole (1991) and Osborne and Rubinstein (1994) for the two usual definitions.

$\{L', R'\}$. Payoffs are $(0,0)$ and $(3,1)$ after One chooses L' and R', respectively. Let a_1 be the first node at which player One moves, a_2 be the second node at which she moves and b be the node at which player Two moves.

A generalized extensive-form game will be described as follows. Let $N = \{1, \ldots n\}$ be the set of players. Let H be a set; the elements of H will be referred to as "information sets". Let $\mathcal{H} = \{H_\alpha : \alpha \in I\}$ be a collection of subsets of H and $\{H^i : i \in N\}$ a partition of H. The interpretation will be that player i is endowed with a collection H^i of information sets and that the elements of $H = \cup_{i=1}^n H^i$ are the information sets of the game. For each $\alpha \in I$, $H_\alpha \subseteq H$ should be interpreted as a "subgame" of H (this interpretation is made precise below).

In the example in Figure 2, $N = \{1, 2\}$, $H = \{a_1, a_2, b\}$, $H^1 = \{a_1, a_2\}$ and $H^2 = \{b\}$. There are three subgames in the example, let $I = \{\alpha_0, \alpha_1, \alpha_2\}$, $H_{\alpha_0} = \{a_1, a_2, b\}$, $H_{\alpha_1} = \{a_2, b\}$, and $H_{\alpha_2} = \{a_2\}$; $\mathcal{H} = \{H_{\alpha_0}, H_{\alpha_1}, H_{\alpha_2}\}$.

I make two assumptions about $\mathcal{H}$, the collection of "subgames." First, that H itself belongs to this class of subgames. Let $\alpha_0 \in I$ satisfy $H = H_{\alpha_0}$. Second, if $\{H_\alpha : \alpha \in I'\} \subseteq \mathcal{H}$ is a subcollection of $\mathcal{H}$ such that any $\alpha, \alpha' \in I'$ satisfy either $H_\alpha \subseteq H_{\alpha'}$ or $H_{\alpha'} \subseteq H_\alpha$, then $\cup_{\alpha \in I'} H_\alpha \in \mathcal{H}$. That is, I assume that $\mathcal{H}$ is closed under increasing unions. For every $\alpha \in I$, $\{H^i : i \in N\}$ induces a partition on H_α: let $H_\alpha^i = H_\alpha \cap H^i$ be the set of player i's information sets in subgame H_α. Note that $\mathcal{H}^i = \{H_\alpha^i : \alpha \in I\}$ is also closed under increasing unions. It is easy to verify that the example in Figure 2 satisfies these assumptions, as does any well-defined game tree.

The players choose actions at each of their information sets. For each $h \in H$, let $A(h)$ be the set of actions available to the player that moves at information set h. Each $A(h)$ is endowed with a Hausdorff topology. The set of all possible actions is denoted by $\mathcal{A} = \cup_{h \in H} A(h)$. Player i's strategy space in subgame $\alpha \in I$ is $S_\alpha^i = \{s : H_\alpha^i \to \mathcal{A} : s(h) \in A(h) \text{ for all } h \in H_\alpha^i\} = \Pi_{h \in H_\alpha} A(h)$. Let $S_\alpha = \times_{i=1}^n S_\alpha^i$. Each player is endowed with preferences over strategy profiles in subgame $\alpha \in I$. These preferences are represented by a collection of payoff functions $u^i_\alpha : S_\alpha \to \mathbf{R}$.

In the example, $A(a_1) = \{L, R\}$, $A(b) = \{l, r\}$ and $A(a_2) = \{L', R'\}$, so that $\mathcal{A} = \{L, L', R, R', l, r\}$. The strategy space for player 1 for the whole game is $S^1_{\alpha_0} = \{LL', LR', RL', RR'\}$—where LL' means that 1 plans to play L at her first decision node, a_1, and then L' at her second decision node, a_2, and so on. The strategy space for player 2 for the whole game is simply $S^2_{\alpha_0} = \{l, r\}$. The strategy spaces for the other subgames are $S^1_{\alpha_1} = \{L', R'\}$, $S^2_{\alpha_1} = \{l, r\}$, $S^1_{\alpha_2} = \{L', R'\}$ and $S^2_{\alpha_2} = \{\emptyset\}$. The choice of $\emptyset$ for player 2 in subgame α_2 formalizes that only 1 makes a choice in this subgame. The players' preferences in each subgame are immediate from Figure 2, $u^1_{\alpha_0}(LL', l) = 2$,$u^2_{\alpha_0}(LL', l) = 0$, $u^1_{\alpha_1}(L', l) = 0$, $u^2_{\alpha_1}(L', l) = 1$, $u^1_{\alpha_2}(L', \emptyset) = 0$, $u^2_{\alpha_2}(L', \emptyset) = 0$, etc.

Definition 1 *A collection of payoff functions $\{u^\alpha_i : i \in N, \alpha \in I\}$ is* ***consistent*** *if, for every $i \in N$ and $\alpha, \alpha' \in I$, whenever $H_{\alpha'} \subseteq H_\alpha$, $s^{-i}_\alpha \in S^{-i}_\alpha$, $s^i_\alpha, \tilde{s}^i_\alpha \in S^i_\alpha$ and $z^i_{\alpha'} \in S^i_{\alpha'}$, the inequalities $u^i_\alpha(\tilde{s}^i_\alpha, s^{-i}_\alpha) \le u^i_\alpha(s^i_\alpha, s^{-i}_\alpha)$ and $u^i_{\alpha'}(s^i_\alpha|_{H^i_{\alpha'}}, s^{-i}_\alpha|_{H^{-i}_{\alpha'}}) \le u^i_{\alpha'}(z^i_{\alpha'}, s^{-i}_\alpha|_{H^{-i}_{\alpha'}})$ imply that $u^i_\alpha(\tilde{s}^i_\alpha, s^{-i}_\alpha) \le u^i_\alpha(s^i_\alpha|_{H^i_\alpha \backslash H^i_{\alpha'}}, z^i_{\alpha'}, s^{-i}_\alpha)$. The collection of payoff functions satisfies* ***continuity*** *if, for all $\alpha \in I$, $i \in N$, and $s^{-i}_\alpha \in S^{-i}_\alpha$, $s^i_\alpha \mapsto u^i_\alpha(s^i_\alpha, s^{-i}_\alpha)$ is an upper semi-continuous function.*

Payoffs are consistent if, given opponents' strategy s^{-i}, whenever s^i_α performs better than $\tilde{s}^i_\alpha$ in subgame $H_\alpha \supset H_{\alpha'}$ and $z^i_{\alpha'}$ performs better than s^i_α in subgame $H_{\alpha'}$, it must be that the combined strategy that follows s^i_α in $H^i_\alpha \backslash H^i_{\alpha'}$ and follows $z^i_{\alpha'}$ in $H^i_{\alpha'}$, cannot perform worse than $\tilde{s}^i_\alpha$.

The payoffs in the example are consistent: Fix the strategy $s^2_{\alpha_0} = l$ by player 2 in subgame α_0, the "whole" game. Given any strategy $s^1_{\alpha_0}$, player 1's payoff is independent of choices in node a_2. In particular, choosing R', the dominant strategy in subgame α_2, does not decrease the payoff to following $s^1_{\alpha_0}$. Now, consider $s^2_{\alpha_0} = r$. The only case where the requirement consistency has bite is for the strategy LL'. In subgame α_0, LL' is preferred by 1 to RL'. But, in subgame α_2, R' is better than L'. Consistency then requires that LR' be preferred to RL' in subgame α_0—which is satisfied by the specified payoffs.

The example illustrates why payoffs in any well-defined game tree are consistent. Given i's strategy s^i_α and opponents' strategies s^{-i}_α, if a subgame α' is not reached then i is indifferent among her choices in this subgame and she cannot do worse by picking something that is better in the subgame. On the other hand, if subgame α' is reached then payoffs will be given by choices in α'. Choosing a better strategy in subgame α' can only improve the payoff to s^i_α.

The definition of a generalized extensive-form game is complicated enough to warrant an enumeration of its components:

Definition 2 *The sextuple* $\Gamma = \{N, H, \{H_\alpha : \alpha \in I\}, \{H^i : i \in N\}, \{A(h) : h \in H\}, \{u^i_\alpha : i \in N, \alpha \in I\}\}$ *is a* ***generalized extensive-form game*** *if:*

- $N = \{1, 2, \ldots n\}$ *is the set of players;*
- H *is a set of information sets;*
- $\{H^i : i \in N\}$ *is a partition of* H*;*
- $\{H_\alpha : \alpha \in I\}$ *is a collection of subsets of* H *that is closed under increasing unions and such that* $H_{\alpha_0} = H$ *for some* $\alpha_0 \in I$*;*
- $\{A(h) : h \in H\}$ *is a collection of action sets, each endowed with a Hausdorff topology and compact;*
- $\{u^\alpha_i : i \in N, \alpha \in I\}$ *is a collection of consistent payoff functions that satisfies continuity.*

For any subgame $\alpha \in I$, Γ induces naturally an extensive-form game Γ_α: let H_α be the set of information sets of Γ_α, let the subgames $\gamma \in I$ with $H_\gamma \subseteq H_\alpha$ be the subgames of Γ_α; and let action sets and payoffs be defined as in Γ. I will use "subgame" to denote both the set H_α and its corresponding extensive-form game Γ_α.

A strategy profile s_α is a ***subgame-perfect Nash equilibrium*** (SPNE) in subgame $\alpha \in I$ if, for every $\gamma \in I$ such that $H_\gamma \subseteq H_\alpha$ and every $i \in N$,

$$s^i_\alpha|_{H_\gamma} \in \text{argmax}_{\tilde{s}^i_\gamma \in S^i_\gamma} u^\gamma(\tilde{s}^i_\gamma, s^{-i}_\alpha|_{H_\gamma}).$$

I shall refer to the SPNE in subgame α_0, the whole game, as simply SPNE. Note that a strategy profile is a SPNE if and only if its restriction to any subgame is a SPNE in that subgame.

Definition 3 *A generalized extensive-form game Γ is an **extensive-form game of strategic complementarities** if $A(h)$ is a complete lattice for all $h \in H$ and if, for all $\alpha \in I$ and $i \in N$, $s^i_\alpha \mapsto u^i_\alpha(s^i_\alpha, s^{-i}_\alpha)$ is quasisupermodular on S^i_α, and u^i_α satisfies the single-crossing condition in $(s^i_\alpha, s^{-i}_\alpha)$.*

It is slightly cumbersome to show that the game in Figure 1, has strategic complements. I leave this for section 4.

2.3 Examples of Generalized Extensive-Form Games

2.3.1 Optional BoS

I shall present the current notation for the "Optional Battle of the Sexes" game from the introduction. Let a_1 be the initial node, b be the node at which player Two moves and a_2 be One's information set after that Two has moved. Then, $H = \{a_1, a_2, b\}$, $H^1 = \{a_1, a_2\}$, $H^2 = \{b\}$. There are two subgames, so $I = \{\alpha_0, \alpha_1\}$, $H_{\alpha_0} = \{a_1, a_2, b\}$ and $H_{\alpha_1} = \{b, a_2\}$. Action spaces are $A(a_1) = \{\text{Yes}, \text{No}\}$, $A(b) = A(a_2) = \{O, B\}$. Strategy spaces are $S^1_{\alpha_0} = \{\text{Yes}O, \text{Yes}B, \text{No}O, \text{No}B\}$, $S^2_{\alpha_0} = S^1_{\alpha_1} = S^2_{\alpha_1} = \{O, B\}$.

2.3.2 Battle of the Sexes in Continuous Time

The game is a Battle of the Sexes played in continuous time. As Anderson (1984) and Simon and Stinchcombe (1989) point out, the map from strategies to outcomes might not be well defined in continuous-time games, which implies that we cannot define the payoffs resulting from different strategy profiles. I will not spell out the details in Simon and Stinchcombe's (1989) model of continuous-time extensive-form games, I only present a simple example (that in fact falls within Simon and Stinchcombe's framework).

To avoid problems with the map from strategies to outcomes, I impose that players must switch infrequently from one action to the other. Time is indexed by $t \in [0, 1]$. Assume that players choose either O or B at time $t = 0$. Their decisions remain fixed for a period $\delta \in (0, 1)$; at time $t = \delta$ they can choose to switch, represented by action 1, or not to change their time 0 choice, represented by action 0. An any posterior time t, players are allowed to choose 1 only if they have chosen 0 in $[t - \delta, t)$. That is, switches are irreversible for a length of time δ. The players' "flow" payoffs are as in BoS in the introduction. If, at time t, they both choose O then One gets a payoff of 2 while Two gets a payoff of 1; when they both choose B, then One gets 1 while Two gets 2. If they choose different actions at time t then they both get a payoff of 0.

For any $t \in [0, 1]$, the events until time t are described by a pair of vectors $h_t = (h_t^1, h_t^2)$, with $h_t^i = (t_1^i, \ldots t_k^i)$ and where t_l^i is the time of the lth switch by player i. By the description above, we must have $t_l^i - t_{l-1}^i \geq \delta$ for $l = 2, \ldots k$ and $t_k^i \leq t$. Player i starts with action O if $t_1^i = 0$, with action B if $t_1^i > 0$. Any feasible h_t is called a time-t history. Let H_t be the set of all time-t histories.

The set of all information sets is $H = \cup_{t \in [0,1]} H_t$. Any history h_t starts a subgame $H_{h_t} = \{h_\tau \in H : t \leq \tau, h_\tau | t = h_t\}$, where $h_\tau | t = h_t$ means that h_τ and h_t coincide on switches before time t. For any history h_t, the actions available to player i are $\{0, 1\}$

or, if she has switched recently (so $t - t_k < \delta$), $\{0\}$. I will show that the collection of information sets is closed under increasing unions. Let $\left\{H_{h_t} : h_t \in \tilde{H}\right\}$ be an increasing collection of subgames and let $\underline{t} = \inf\left\{t : h_t \in \tilde{H}\right\}$. Then, all histories h_t coincide on switches up to $\underline{t}$, let $h_{\underline{t}}$ be this common history. It is immediate that $H_{h_{\underline{t}}} = \cup_{h_t \in \tilde{H}} H_{h_t}$. Similarly, the collection of information sets is closed under intersections.

Strategies are maps $h_t \mapsto s^i(h_t) \in A^i(h_t)$, where $A^i(h_t)$ is $\{0\}$ if the last switch in h_t was later than $t - \delta$ and $A^i(h_t)$ is $\{0, 1\}$ else. A pair of strategies define, recursively, a finite collection of switches. A finite collection of switches gives, through the definition of flow payoffs above, the payoff associated to the strategy profile. Additivity of payoffs (from flow payoffs) implies immediately that payoffs are consistent.

3 The Extended Best-Response Correspondence and Strategic Complementarities

3.1 Main Results

In this paper I shall focus on subgame-perfect equilibria. In order to keep track of the best responses to opponents' strategies in each subgame, I need to introduce the set $\mathcal{S}^i = \left\{\mathsf{s}^i \in \Pi_{\alpha \in I} S^i_\alpha : \mathsf{s}^i_{\alpha_0}|_{H^i_\alpha} = \mathsf{s}^i_\alpha\right\}$. This is the set of lists $\mathsf{s}^i \in \Pi_{\alpha \in I} S^i_\alpha$ so that the component $\mathsf{s}^i_\alpha \in S^i_\alpha$ that corresponds to subgame H_α coincides with the restriction of $\mathsf{s}^i_{\alpha_0} \in S^i_{\alpha_0}$—the strategy for the whole game—to subgame H_α. Let $\mathcal{S} = \times_{i=1}^n \mathcal{S}^i$. For the example in Figure 2, recall that $S^1_{\alpha_0} = \{LL', LR', RL', RR'\}$, $S^1_{\alpha_1} = \{L', R'\}$, and $S^1_{\alpha_2} = \{L', R'\}$. So,

$$\mathcal{S}^1 = \{(LL', L', L'), (LR', R', R'), (RL', L', L'), (RR', R', R')\},$$

which is really the same set as $S^1_{\alpha_0}$, $\mathcal{S}^1$ is an accounting device. In general S^i and $\mathcal{S}^i$ are isomorphic: identify $\mathsf{s}^i \in \mathcal{S}^i$ with $\mathsf{s}^i_{\alpha_0} \in S^i$. In the rest of the paper I will frequently identify S and $\mathcal{S}$.

Definition 4 *Player i's* ***extended best-response correspondence*** $\beta^i : \mathcal{S} \twoheadrightarrow \mathcal{S}^i$ *is defined by:*

$$\beta^i(\mathsf{s}) = \left\{ \mathsf{s}^i \in \mathcal{S}^i : u^i_\alpha(\mathsf{s}^i_\alpha, \mathsf{s}^{-i}_\alpha) \geq u^i_\alpha(\hat{s}^i_\alpha, \mathsf{s}^{-i}_\alpha) \;\; \textit{for all } \hat{s}^i_\alpha \in S^i_\alpha, \;\; \textit{for all } \alpha \in I \right\}.$$

The game's extended best-response correspondence is $\beta : \mathcal{S} \twoheadrightarrow \mathcal{S}$*, defined as* $\beta(\mathsf{s}) = \times_{i \in N} \beta_i(s)$*.*

Player i's extended best-response correspondence assigns a strategy that is a best response *in each subgame* to her opponents' strategy. A game Γ's SPNE can be analyzed by means of its extended best-response correspondence β. Lemma 1 shows the usefulness of the extended best-response construction. The construction of β shows immediately why Lemma 1 is true, so the lemma's proof is omitted.

Lemma 1 *The set of SPNE of a generalized extensive-form game equals the fixed points of its extended best-response correspondence.*

Lemma 2 shows that β is not a vacuous construction. The idea behind its proof is simple. Given opponents' strategies $s^{-i}_{\alpha_0}$, if $s^i_{\alpha_0}$ is a best response for player i in the whole game H_{α_0}, then $s^i_{\alpha_0}$ should prescribe a best response for subgames that are reached under $s^{-i}_{\alpha_0}$. Also, i is indifferent between strategies on subgames that are not reached. Modifying $s^i_{\alpha_0}$ to play a best response to $s^{-i}_{\alpha_0}$ also on non-reached subgames yields, by consistency of payoffs, a strategy that is still a best response to $s^{-i}_{\alpha_0}$ in the original game H_{α_0}. Repeating this operation "subgame by subgame" we can obtain an element in $\beta(\mathsf{s})$.

The reasoning "subgame by subgame" suggests a proof by induction. Even in simple games (like infinitely repeated bimatrix games) the set of subgames is uncountable, so a proof by induction is not possible. The proper tool turns out—expectedly—to be Zorn's Lemma.

Lemma 2 *For all* $\mathsf{s} \in \mathcal{S}$, $\beta(\mathsf{s})$ *is not empty.*

Proof: Let $\mathsf{s} \in \mathcal{S}$ and fix $i \in N$. For any $\alpha \in I$, let $\beta^i_\alpha(\mathsf{s}^{-i}) = \text{argmax}_{s^i \in S^i_\alpha} u^i_\alpha(s^i, \mathsf{s}^{-i}_\alpha)$. Tychonoff's Theorem implies that S^i_α is compact, so $\beta^i_\alpha(\mathsf{s}^{-i})$ is nonempty because S^i_α is compact and payoffs are upper semi-continuous in the player's own strategy. Let $\Omega = \{(s^i_\alpha, H^i_\alpha) : \alpha \in I, s^i_\alpha \in \beta^i_\alpha(\mathsf{s}^{-i})\}$ be the set of pairs of best responses and subgames. Order Ω by $\preceq$, where $(s^i_{\alpha'}, H^i_{\alpha'}) \preceq (s^i_\alpha, H^i_\alpha)$ if and only if $H^i_{\alpha'} \subseteq H^i_\alpha$ and $s^i_\alpha|_{H^i_{\alpha'}} = s^i_{\alpha'}$. It is immediate to verify that $\preceq$ is a partial order on Ω.

Let $\tilde{\Omega} \subseteq \Omega$ be a linearly ordered subset of Ω. Say $\tilde{I} \subseteq I$ is such that $\tilde{\Omega} = \left\{(s^i_\alpha, H^i_\alpha) \in \Omega : \alpha \in \tilde{I}\right\}$. Let $\hat{H} = \cup_{\alpha \in \tilde{I}} H^i_\alpha$, since $\mathcal{H}^i$ is closed under increasing unions $\hat{H} \in \mathcal{H}^i$. Let $\gamma \in I$ satisfy $H^i_\gamma = \hat{H}$. For any $h \in H_\gamma$, there is $\alpha \in \tilde{I}$ such that $h \in H_\alpha$; construct $s^i_\gamma \in S^i_\gamma$ by setting $s^i_\gamma(h) = s^i_\alpha(h)$. Since $\tilde{\Omega}$ is linearly ordered, s^i_γ is well defined.

I will show that (H^i_γ, s^i_γ) is an upper bound on $\tilde{\Omega}$ in Ω. Clearly, $H^i_\alpha \subseteq H^i_\gamma$ and $s^i_\gamma|_{H^i_\alpha} = s^i_\alpha$ for all $\alpha \in \tilde{I}$. Let $\overline{s}^i_\gamma \in \beta^i_\gamma(\mathsf{s}^{-i})$ and let $\left\{s^i_\gamma(\alpha)\right\}_{\alpha \in \tilde{I}}$ be the net in S^i_γ obtained by $s^i_\gamma(\alpha)|_{H^i_\alpha} = s^i_\gamma|_{H^i_\alpha}$, $s^i_\gamma(\alpha)|_{H^i_\gamma \setminus H^i_\alpha} = \overline{s}^i_\gamma|_{H^i_\gamma \setminus H^i_\alpha}$, and directing $\tilde{I}$ by set inclusion. Note that $s^i_\gamma(\alpha) \to s^i_\gamma$ in the product topology. Fix any $\tilde{s}^i_\gamma \in S^i_\gamma$. Then $\overline{s}^i_\gamma \in \beta^i_\gamma(\mathsf{s}^{-i})$ implies that $u^i_\gamma(\tilde{s}^i_\gamma, \mathsf{s}^{-i}_\gamma) \leq u^i_\gamma(\overline{s}^i_\gamma, \mathsf{s}^{-i}_\gamma)$. Now, $s^i_\gamma|_{H^i_\alpha} = s^i_\alpha \in \beta^i_\alpha(\mathsf{s}^{-i})$ implies that $u^i_\alpha(\overline{s}^i_\gamma|_{H^i_\alpha}, \mathsf{s}^{-i}_\alpha) \leq u^i_\alpha(s^i_\gamma|_{H^i_\alpha}, \mathsf{s}^{-i}_\alpha)$. By consistency of payoffs, then, for any $\alpha \in \tilde{I}$, $u^i_\gamma(\tilde{s}^i_\gamma, \mathsf{s}^{-i}_\gamma) \leq u^i_\gamma(s^i_\gamma(\alpha), \mathsf{s}^{-i}_\gamma)$. But then $u^i_\gamma(\tilde{s}^i_\gamma, \mathsf{s}^{-i}_\gamma) \leq u^i_\gamma(s^i_\gamma, \mathsf{s}^{-i}_\gamma)$, as $s^i_\gamma(\alpha) \to s^i_\gamma$ and payoffs are upper semi-continuous. This shows that $(H^i_\gamma, s^i_\gamma) \in \Omega$, so (H^i_γ, s^i_γ) is an upper bound on $\tilde{\Omega}$.

The linearly ordered set $\tilde{\Omega}$ was arbitrary. By Zorn's lemma there is a maximal element, say $(H^{*i}_\alpha, s^{*i}_\alpha)$, of Ω. Suppose $H^{*i}_\alpha \neq H^i_{\alpha_0}$. Let $\overline{s}^i_{\alpha_0} \in \beta^i_{\alpha_0}(\mathsf{s}^{-i})$. Define $s^i_{\alpha_0} \in S^i_{\alpha_0}$ by $s^i_{\alpha_0}|_{H^i_{\alpha_0} \setminus H^{*i}_\alpha} = \overline{s}^i_{\alpha_0}|_{H^i_{\alpha_0} \setminus H^{*i}_\alpha}$ and $s^i_{\alpha_0}|_{H^{*i}_\alpha} = s^{*i}_\alpha$. Now, $(H^{*i}_\alpha, s^{*i}_\alpha) \in \Omega$ implies that $s^{*i}_\alpha \in \beta^i_\alpha(\mathsf{s}^{-i})$. Let $\tilde{s}^i_{\alpha_0} \in S^i_{\alpha_0}$, then $u^i_{\alpha_0}(\tilde{s}^i_{\alpha_0}, \mathsf{s}^{-i}_{\alpha_0}) \leq u^i_{\alpha_0}(\overline{s}^i_{\alpha_0}, \mathsf{s}^{-i}_{\alpha_0})$ and $u^i_\alpha(\overline{s}^i_{\alpha_0}|_{H^{*i}_\alpha}, \mathsf{s}^{-i}_\alpha) \leq u^i_\alpha(s^{*i}_{\alpha_0}|_{H^{*i}_\alpha}, \mathsf{s}^{-i}_\alpha)$. By consistency of payoffs, $u^i_{\alpha_0}(\tilde{s}^i_{\alpha_0}, \mathsf{s}^{-i}_{\alpha_0}) \leq u^i_{\alpha_0}(s^i_{\alpha_0}, \mathsf{s}^{-i}_{\alpha_0})$. Hence, $s^i_{\alpha_0} \in \beta^i_{\alpha_0}(\mathsf{s}^{-i})$, so that $(H^i_{\alpha_0}, s^i_{\alpha_0}) \in \Omega$ and $(H^{*i}_\alpha, s^{*i}_\alpha) \preceq (H^i_{\alpha_0}, s^i_{\alpha_0})$. But $(H^{*i}_\alpha, s^{*i}_\alpha) \neq (H^i_{\alpha_0}, s^i_{\alpha_0})$. Contradiction, since $(H^{*i}_\alpha, s^{*i}_\alpha)$ is maximal.

Construct $\mathsf{s}^{*i} \in \mathcal{S}^i$ by setting $\mathsf{s}^{*i}_\alpha = s^{*i}_{\alpha_0}|_{H^i_\alpha}$ for all $\alpha \in I$. It is then immediate that $\mathsf{s}^{*i} \in \beta^i(\mathsf{s})$ since, by construction, for all $\alpha \in I$, $\mathsf{s}^{*i}_\alpha = s^{*i}_\alpha \in \beta^i_\alpha(\mathsf{s}^{-i})$. ■

Extended best-response correspondences translate the problem of finding SPNE to a fixed-point problem. By adding the assumption of strategic complementarities, fixed points are obtained by a version of Tarski's Fixed Point Theorem and the sets of SPNE can be analyzed by "lattice programming" techniques. Potentially, though, extended best-response correspondences are useful in other classes of extensive-form games as well.

Theorem 1 *If Γ is an extensive-form game of strategic complementarities, then its SPNE form a non-empty, complete lattice.*

Proof: I need to show that β is monotone increasing in the strong set order and takes non-empty, closed values in order to apply Zhou's (1994) version of Tarski's fixed point theorem. First I show that β is monotone increasing in the strong set order. Let $\mathsf{s}, \mathsf{z} \in \mathcal{S}$ with $\mathsf{s} \prec \mathsf{z}$. Let $\mathsf{s}' \in \beta(\mathsf{s})$ and $\mathsf{z}' \in \beta(\mathsf{z})$. By Theorem 4 in Milgrom and Shannon (1994), for every $\alpha \in I$, $\mathsf{s}'^{\,i}_\alpha \vee \mathsf{z}'^{\,i}_\alpha \in \mathrm{argmax}_{s \in S^i_\alpha} u^i_\alpha(s, \mathsf{z}^{-i}_\alpha)$ and $\mathsf{s}'^{\,i}_\alpha \wedge \mathsf{z}'^{\,i}_\alpha \in \mathrm{argmax}_{s \in S^i_\alpha} u^\alpha_i(s, \mathsf{s}_{-i,\alpha})$. Hence, $\mathsf{s}' \vee \mathsf{z}' \in \beta(\mathsf{z})$ and $\mathsf{s}' \wedge \mathsf{z}' \in \beta(\mathsf{s})$, proving that β is increasing in the strong set order.

That β takes closed values is an immediate consequence of upper semi-continuity of payoffs in each subgame. By Lemma 2, β takes non-empty values. Hence, by Zhou's version of Tarski's fixed point theorem, the set of fixed points of β is a complete lattice. Lemma 1 implies that the set of SPNE is a complete lattice. ■

Theorem 1 implies that there is a smallest and a largest SPNE of any extensive-form game of strategic complementarities. Note that the subgames of any extensive-form game of strategic complements are also extensive-form games of strategic complements. By Theorem 1, then, each subgame has a smallest and a largest SPNE strategy profile. It turns out that the extremal SPNE of any subgame are obtained from the extremal

SPNE of the whole game. [4]

The collection of subgames $\{H_\alpha : \alpha \in I\}$ is ***closed under intersections*** if for any $\alpha, \alpha' \in I$, there is $\xi \in I$ such that $H_\xi = H_\alpha \cap H_{\alpha'}$. Any well-defined game tree has subgames that are closed under intersections (in fact subgames are either nested or disjoint, so they are trivially closed under intersections).

Theorem 2 *Let Γ be an extensive-form game of strategic complementarities with subgames that are closed under intersections; let $\underline{s}$ be its smallest SPNE and $\overline{s}$ its largest SPNE. If H_α with $\alpha \in I$ is any subgame, then $\underline{s}|_{H_\alpha}$ and $\overline{s}|_{H_\alpha}$ are, respectively, the smallest and largest SPNE of the extensive-form game corresponding to H_α.*

Proof: Suppose, by way of contradiction, that there is a subgame H_α with a smallest SPNE strategy profile s_α that is not equal to $\underline{s}|_{H_\alpha}$. Let $\tilde{\mathcal{S}} = \{\hat{\mathsf{s}} \in \mathcal{S} : \hat{\mathsf{s}}_\alpha = s_\alpha\}$, by repeating the arguments above we obtain that $\tilde{\mathcal{S}}$ is a complete lattice. Let $\tilde{\beta} : \tilde{\mathcal{S}} \twoheadrightarrow \tilde{\mathcal{S}}$ be defined by $\tilde{\beta}(\mathsf{s}) =$

$$\left\{\hat{\mathsf{s}} \in \tilde{\mathcal{S}} : u^i_\gamma(s'^{\,i}_\gamma, \mathsf{s}^{-i}_\gamma) \leq u^i_\gamma(\hat{\mathsf{s}}^i_\gamma, \mathsf{s}^{-i}_\gamma) \text{ for all } s'^{\,i}_\gamma \in S^i_\gamma, s'^{\,i}_\gamma|_{H_\alpha} = s^i_\alpha, i \in N, \text{ and } \gamma \in I, H_\gamma \nsubseteq H_\alpha\right\}.$$

That $\tilde{\beta}$ is monotone increasing in the strong set order and closed-valued follows from arguments similar to those proving that β is monotone increasing and closed-valued. That $\tilde{\beta}$ has non-empty values can be proved by following the steps in the proof of Lemma 2, and restricting the optimizing strategies to equal s^i_α on information sets that also belong to H_α.

By Zhou's version of Tarski's Theorem, there is a fixed point $\mathsf{s} \in \tilde{\mathcal{S}}$ of $\tilde{\beta}$. I claim that this is a SPNE of the whole game H_{α_0}. Fix $i \in N$. For any $\gamma \in I$ with $H_\gamma \subseteq H_\alpha$, $\mathsf{s}_{\alpha 0}|_{H_\gamma} = s_\alpha|_{H_\gamma}$. But $\mathsf{s}^i_{\alpha 0}|_{H_\gamma} \in \beta^i_\gamma(\mathsf{s})$ since H_γ is a subgame of H_α and s_α is a SPNE in H_α. Let $\gamma \in I$ with $H_\gamma \nsubseteq H_\alpha$. Γ has subgames that are closed under intersections, hence

[4] This has important consequences. It can be seen that, in multi-stage games, the extremal equilibria are Markov-Perfect.

there is $\xi \in I$ with $H_\xi = H_\gamma \cap H_\alpha$. Since H_ξ is a subgame of H_α, $\mathsf{s}_\xi = s_\alpha|_{H_\xi}$ so $\mathsf{s}_\xi \in \beta^i_\xi(\mathsf{s})$ because s_α is a SPNE in subgame H_α. Then, $\mathsf{s}^i_\gamma \in \beta^i_\gamma(\mathsf{s})$ because if there is $\hat{s}^i_\gamma \in S^i_\gamma$ with $u^i_\gamma(\mathsf{s}^i_\gamma, \mathsf{s}^{-i}_\gamma) < u^i_\gamma(\hat{s}^i_\gamma, \mathsf{s}^{-i}_\gamma)$ it must be that $\hat{s}^i_\gamma$ and s^i_γ differ on the subgame $H_\xi = H_\gamma \cap H_\alpha$. But, using consistency of payoffs,

$$u^i_\gamma(\mathsf{s}^i_\gamma, \mathsf{s}^{-i}_\gamma) < u^i_\gamma(\hat{s}^i_\gamma, \mathsf{s}^{-i}_\gamma) \leq u^i_\gamma(\hat{s}^i_\gamma|_{H_\gamma \setminus H_\alpha}, \mathsf{s}^i_\xi, \mathsf{s}^{-i}_\gamma),$$

impossible since $\mathsf{s} \in \tilde{\beta}(\mathsf{s})$. Hence, $\mathsf{s}^i_\gamma \in \beta^i_\gamma(\mathsf{s})$ for any $\gamma \in I$ so s is a SPNE.

By Theorem 1 there is a SPNE $\tilde{s} = \underline{s} \wedge \mathsf{s}_{\alpha_0}$ because the set of SPNE is a lattice. Since $\underline{s}$ and s_{α_0} differ on H_α, $\tilde{s}$ is smaller than $\underline{s}$, a contradiction. ∎

Once the structure of complementarities is present, comparative statics results for the extremal SPNE follow from well-known results in Monotone Comparative Statics. [5]

Definition 5 *Let T be a partially ordered set. The collection $\Gamma(t) : t \in T$ is an **increasing family of extensive-form games** if, for any $t \in T$, $\Gamma(t) = \{N, H, \{H_\alpha : \alpha \in I\}, \{H_i : i \in N\}, \{A_t(h) : h \in H\}, \{u^i_{\alpha\, t} : i \in N, \alpha \in I\}\}$ is an extensive-form game of strategic complementarities; if for all $h \in H$, $A_t(h)$ is increasing in the strong set order in t and if, for all $i \in N$ $\alpha \in I$, $u^i_{\alpha\, t}$ satisfies the single-crossing condition in (s^i_α, t).*

Theorem 3 *Let $\{\Gamma(t) : t \in T\}$ be an increasing family of extensive-form games. Let $t, t' \in T$ with $t \preceq t'$. The smallest (largest) SPNE of $\Gamma(t)$ is smaller, as an element of S, than the smallest (largest) SPNE of $\Gamma(t')$.*

Proof: Let β_t and $\beta_{t'}$ be the extended best-response correspondences of $\Gamma(t)$ and $\Gamma(t')$, respectively. An argument similar to the proof that the extended best-response function is monotone increasing in the proof of Theorem 1 establishes that, for any $\mathsf{s} \in \mathcal{S}$, $\beta_t(\mathsf{s})$ is smaller than $\beta_{t'}(\mathsf{s})$ in the strong set order. The result then follows from Theorem 2.5.2 in Topkis (1998). ∎

[5] See Echenique (2001b) for comparative statics results for non-extremal equilibria.

4 Complementarities are restrictive

How common is the existence of complementarities in extensive-form games? There are two answers to this question. First, one can argue that the order on strategies is not part of the description of a game, so one has a "degree of freedom" in checking for complementarities, one can try to find an order on strategies such that a game has complementarities. So, how often can one find an order on strategies such that a game has complementarities? Very often—I shall not expand on why here, but it is a direct application of the characterization in Echenique (2001a); Echenique's (2001a) results require, though, that one knows first the number of equilibria of the game, so it does not provide sufficient conditions for existence, or for comparative statics.

Second, given games with some kind of "natural" order for which heuristically one *should* get complementarities—for example a dynamic variant of a static game with complementarities—the answer is negative. In the discussion below I shall give examples of games with and without extensive-form complementarities, I believe these examples explain where the problems arise.

The situation contrasts with the study of Markov Perfect equilibria in stochastic games with complementarities. For example, Curtat (1996) (implicitly also Amir (1989) and Amir (1996)) imposes supermodularity conditions on payoffs, and is able to obtain results on the set of Markov Perfect equilibria using lattice-theoretic tools. The reason is that Markov-Perfect equilibria limit the strategic interaction over time enough so that the effect of complementarities is preserved in dynamic contexts—this will, I hope, be clear from the examples below.

1. Optional BoS in the introduction is a game of strategic complementarities. Optional BoS II in Figure 3 cannot be made into a game of strategic complementarities. Say that Yes is larger than No and repeat the argument from the introduction: An increase

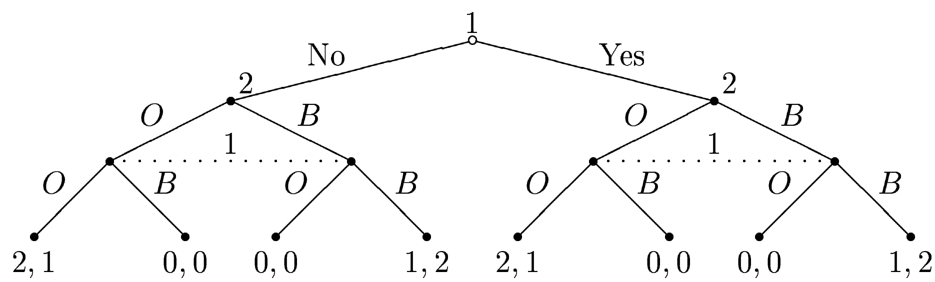

Figure 3: Optional BoS II

in One's strategy from No-OO to Yes-OO makes Two shift from B to O in the game following Yes. Strategic complementarities requires then that the action O is larger than B at this information set. But this gives a decreasing response when we shift from No-OB to Yes-OB. The solution in the introduction was to change the order on $\{\text{Yes}, \text{No}\}$. But here this clearly gives rise to the same situation in the subgame following No. [6]

The problem is that, off the path specified by a strategy profile, players are indifferent between actions. This indifference makes the single-crossing property kick in. If the information set h is off the path of a strategy profile (s^i, s^{-i}) then player i is indifferent between strategies that differ on $A(h)$. Now, if s'^{-i} is a larger strategy, and h is on the path of (s^i, s'^{-i}), then it must be that payoffs are such that, if $a, a' \in A(h)$ and a is smaller than a', then a is preferred to a'. That is, preferences have to coincide with the order on actions for every strategy profile that has h "on its path".

2. The example in item 1 is a dynamic game that intuitively should satisfy definition 3, but fails to do so. Theorems 1, 2 and 3 depend on the property that best responses are monotone increasing, and not directly on the supermodularity/single-crossing assumptions on payoffs in definition 3—these assumptions are merely sufficient for monotone (extended) best responses. I shall show that, even in simple dynamic games, while "per-

[6] Optional BoS II shows that the property of having complementarities is not robust to the addition of an irrelevant move. This is true also in normal-form games, not really the reason why complementarities are especially restrictive in extensive-form games.

period payoffs" are supermodular, (extended) best responses are not monotone increasing, there is no largest SPNE, and the set of SPNE is not a lattice.

Consider the matrix game in Figure 4. It is easy to check that this game has strategic complementarities when each player's strategy set is ordered by $\alpha < \beta < \gamma$. Indeed, as each player's strategy set is totally ordered, payoffs are quasisupermodular in the player's own strategies. The single-crossing property holds too, note that for player "Row" the single-crossing property only kicks in when "Column" plays β and Row compares the payoff of β with that of the smaller strategy α—the payoff to Row of β is higher, and it remains higher when Column increases her strategy from β to γ, so Row's payoffs satisfy the single-crossing property. Similarly with Column's payoffs.

Now suppose that Row and Column play the twice-repeated version of the matrix game. Suppose that they discount per-period payoffs with a discount factor $\delta \geq 1/2$. I shall show that best responses in the twice-repeated version of the game are not monotone increasing, that there is no largest SPNE, and that there are non-monotone SPNE outcomes.

Consider the following strategy for Column: "play γ in period 1, play α in period 2 if the outcome in period 1 was (γ, γ), play β in period 2 if not." A best-response to this strategy by Row is to do the same, i.e. "play γ in period 1, play α in period 2 if the outcome in period 1 was (γ, γ), play β in period 2 if not." Suppose now that Column increases her strategy to "play γ in period 1, play β in period 2 regardless of the period-1

	α	β	γ
α	$3, 3$	$0, 1$	$0, 0$
β	$1, 0$	$1, 1$	$3, 0$
γ	$0, 0$	$0, 3$	$2, 2$

Figure 4: A game with complementarities.

outcome." Then, any best-response by Row involves playing β in period 1—so no best response to the increase in Column's strategy involves an increase in Row's strategy.

One SPNE *outcome* is $((\gamma,\gamma),(\alpha,\alpha))$—meaning that (γ,γ) is the outcome the first time the game is played, and (α,α) is the outcome the second time. Also, $((\beta,\beta),(\beta,\beta))$ is a SPNE outcome. Now, $((\gamma,\gamma),(\alpha,\alpha))$ and $((\beta,\beta),(\beta,\beta))$ are unordered as vectors in $\{\alpha,\beta,\gamma\}^4$, and there are no larger SPNE outcomes, the vectors $((\gamma,\gamma),(\beta,\beta))$ and $((\gamma,\gamma),(\gamma,\gamma))$ are larger, but they cannot be the result of a SPNE. So there is no largest SPNE outcome, which implies that there is no largest SPNE, and that the set of SPNEs is not a lattice.

This example is important because it gets quickly hard to check that a game satisfies definition 3, so it would help if it were enough to check complementarities on per-period payoffs. Then, even if definition 3 is hard to satisfy, there is little hope that it can be weakened in a fruitful way, as any such weakening should reasonably imply that the example above is a game with strategic complementarities.

Contrast this conclusion with the result on Markov strategies in stochastic games (Curtat 1996). In this example, Markov strategies are constant, and independent of history, so the results for stochastic games here are really for the open-loop equilibria of the game. If we allow for more strategic interactions than Markov strategies allow, then the results break down.

3. The problem in the example in item 2 is that complementarities are endogenous. I now explain in more detail how this problem arises.

Suppose there are two players in a two-stage game. Each player i chooses, in time t, a real number a_t^i, the players' choices in each time period are simultaneous. Suppose that player i's payoff from outcome $(a_1^1,a_1^2,a_2^1,a_2^2)$ is $u^i(a_1^1,a_1^2,a_2^1,a_2^2)$, suppose that u^i has the single-crossing property for all possible pairs of its arguments. Note that, if choices were simultaneous, this would be a game with strategic complementarities. Now lets look at the backward induction solution to this game. From e.g. Milgrom and Shannon's (1994) results, there is a Nash equilibrium of each second-stage subgame that is monotone increasing in the first-period choices. That is, let $(a_2^1(a_1^1, a_1^2), a_2^2(a_1^1, a_1^2))$ be a Nash equilibrium in the (a_1^1, a_1^2)-subgame, and $(a_2^1(a_1^1, a_1^2), a_2^2(a_1^1, a_1^2))$ increase as (a_1^1, a_1^2) increases.

It is now easy to see that the induced first-period game may not have complementarities. Let $a_1^1 < \overline{a}_1^1$, $a_1^2 < \overline{a}_1^2$, and suppose that

$$u^i(a_1^1, a_1^2, a_2^1(a_1^1, a_1^2), a_2^2(a_1^1, a_1^2)) \leq u^i(\overline{a}_1^1, a_1^2, a_2^1(\overline{a}_1^1, a_1^2), a_2^2(\overline{a}_1^1, a_1^2)).$$

To obtain single-crossing in the induced first-period payoffs, we need

$$u^i(a_1^1, \overline{a}_1^2, a_2^1(a_1^1, \overline{a}_1^2), a_2^2(a_1^1, \overline{a}_1^2)) \leq u^1(\overline{a}_1^1, \overline{a}_1^2, a_2^1(\overline{a}_1^1, \overline{a}_1^2), a_2^2(\overline{a}_1^1, \overline{a}_1^2)).$$

But the single-crossing assumptions on u_1^i only guarantee that

$$u^i(a_1^1, \overline{a}_1^2, a_2^1(a_1^1, a_1^2), a_2^2(a_1^1, a_1^2)) \leq u^1(\overline{a}_1^1, \overline{a}_1^2, a_2^1(\overline{a}_1^1, a_1^2), a_2^2(\overline{a}_1^1, a_1^2)),$$

and it is easy to generate examples where that is not enough for single crossing the the first-period payoffs. [7]

The problem is that existence of complementarities in period 1 depends on the period-two equilibrium, so static complementarities assumptions are insufficient to guarantee extensive-form complementarities. Definition 3 puts enough structure on across-subgames complementarities so that the induced first-period payoffs do have complementarities. It would be very nice to know if there are weaker conditions that guarantee this, but I think it is very unlikely that there are any.

4. Finally, I will show that the game in Figure 2 is a game of strategic complementarities. Order the actions so that L is larger than R, l is larger than r and L' is larger that R'. Consider Figure 5. The two matrices to the right show the payoffs to player One for each of the eight possible strategy profiles in this game.

[7] For example, $u^i(a_1^1, a_1^2, a_2^1, a_2^2) = a_1^1 a_1^2 a_2^1 a_2^2 + a_1^1(1 - a_1^1) + a_2^1(1 - a_2^1)$ works if we restrict $a_t^i \geq 0$.

s^2:

	R	L
L'	x	$x \vee y$
R'	$x \wedge y$	y

r:

	R	L
L'	0	2
R'	3	2

l:

	R	L
L'	0	2
R'	0	2

Figure 5: Payoffs to Player One in the game in Figure 2

Fix Two's strategy $s^2 = r$. We need to check that $s^1 \mapsto u^1(s^1, r)$ is quasisupermodular. Here, $u^1(RL', r) = 0 < u^1(RL' \wedge LR', r) = u^1(RR', r) = 3$, so the requirement in the definition of quasisupermodularity is vacuous. Fix $s^2 = l$, then $u^1(RL' \wedge LR', l) = u^1(RR', l) = 0$ and $u^1(RL', l) = 0$ so we need $u^1(LR', l) \leq u^1(RL' \vee LR', l)$. Which is satisfied since $u^1(RL' \vee LR', l) = u^1(LL', l) = 2$ and $u^1(LR', l) = 2$. Now, we need to check that the players' payoffs satisfy the single-crossing property. Note from Figure 5 that, when $s^2 = r$, no increase in a strategy by player One is profitable. Since $r < l$ the single-crossing property in player One's payoffs is satisfied vacuously.

s^2:

	R	L
L'	0	0
R'	-1	0

Figure 6: Gain from increasing Two's strategy: $u^2(s^1, l) - u^2(s^1, r)$.

To see that Two's payoffs satisfy the single-crossing property, consider Figure 6. The figure shows the gain to Two $u^2(s^1, l) - u^2(s^1, r)$ from increasing his strategy from r to l. It is seen directly from the figure that whenever $u^2(s^1, l) - u^2(s^1, r) \geq 0$ and $s^1 < s^{1\prime}$ then $u^2(s^{1\prime}, l) - u^2(s^{1\prime}, r) \geq 0$. This establishes that the game in Figure 2 is a game of strategic complementarities.

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